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ON PERTURBATIONS OF PLANE CRACKS

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Abstract—Asymptotic analysis is presented for elasticity problems concerning out-of-plane perturbations of plane cracks. Some of the subtleties are first illustrated through consideration of two-dimensional problems; we derive formulae that generalize slightly those already available. Solution of the more difficult three-dimensional problem is facilitated through novel use of an integral identity. The asymptotic formulae that are developed for the stress intensity factors are more flexible than those available previously, in that no special system of coordinates based on the perturbed crack edge is employed.

1. INTRODUCTION

The present paper deals with two- and three-dimensional problems of linear elasticity in infinite elastic media containing slightly nonplanar cracks. We consider a smooth perturbation of the crack surface and obtain asymptotic formulae for the stress-intensity factors. For the case of two dimensions our results generalize formulae derived by Cotterell and Rice (1980). For the case of three dimensions we present asymptotic formulae for the stress-intensity factors (for the case of the out-of-plane perturbation of the crack) and give analysis which may be useful for the explanation of apparent inconsistencies between results of Gao (1992), Xu *et al.* (1994) and Ball and Larralde (1995).

Asymptotic analysis of the stress intensity factors for problems relating to deflection of the crack trajectory can be used for the prediction of crack propagation in an inhomogeneous elastic medium (for a two-dimensional case see the work of Movchan *et al.*, 1991; three-dimensional numerical simulations were performed by Gao and Rice, 1989; Bower and Ortiz, 1990; Xu *et al.*, 1994). It turns out that the first-order asymptotic approximation of the stress intensity factors requires the two term expansion of the stress field near the unperturbed crack tip as well as two terms of the expansion of the Bueckner weight functions (see Bueckner, 1987).

As a matter of motivation we analyze several elementary examples.

1.1. *A rigid shift of a semi-infinite crack*

First, consider the perturbation of stress-intensity factors due to rigid shift of a semi-infinite crack

$$S_0 = \{\mathbf{x} : x_2 = 0, x_1 < 0\}. \tag{1.1}$$

The resulting crack after the transformation is

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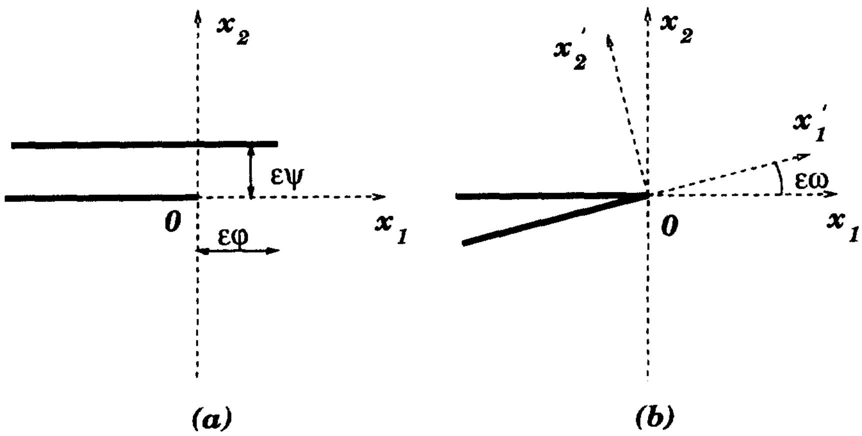

Fig. 1. Rigid shift and rotation of a semi-infinite crack.

$$S_\varepsilon = \{\mathbf{x} : x_2 = \varepsilon\psi, x_1 < \varepsilon\varphi\} \tag{1.2}$$

where $0 < \varepsilon \ll 1$ (see Fig. 1(a)).

Consider a stress field in a homogeneous plane. The equilibrium equations can be written in the form

$$\frac{\partial \sigma_{i1}^{(\mathrm{nc})}}{\partial x_1} + \frac{\partial \sigma_{i2}^{(\mathrm{nc})}}{\partial x_2} + g_i^{(\mathrm{nc})} = 0, \quad i = 1, 2.$$

The superscript (nc) is for "no crack", and it is assumed that the support of $g_i^{(\mathrm{nc})}$ does not intersect the surface where the crack will be located. Introduction of the crack S_0 generates additional stresses σ_{ij} which must cancel the tractions associated with $\sigma^{(\mathrm{nc})}$; thus,

$$\sigma_{12}(x_1, \pm 0) = -\sigma_{12}^{(\mathrm{nc})}(x_1, 0), \quad \sigma_{22}(x_1, \pm 0) = -\sigma_{22}^{(\mathrm{nc})}(x_1, 0), \quad \text{when } x_1 < 0. \tag{1.3}$$

The crack face weight function for the semi-infinite crack (1.1) is given by

$$h(x_1) = \sqrt{\frac{2}{-\pi x_1}}, \quad x_1 < 0. \tag{1.4}$$

The following relations (see Sih and Liebowitz, 1968) hold for the traction vector ahead of the crack

$$\begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} \sim \frac{1}{\sqrt{2\pi x_1}} \begin{pmatrix} K_{\mathrm{II}} \\ K_{\mathrm{I}} \end{pmatrix} + \sqrt{x_1} \begin{pmatrix} A_{\mathrm{II}} \\ A_{\mathrm{I}} \end{pmatrix}, \tag{1.5}$$

where

$$\begin{pmatrix} K_{\mathrm{II}} \\ K_{\mathrm{I}} \end{pmatrix} = \int_{-\infty}^{0} h(x_1) \begin{pmatrix} \sigma_{12}^{(\mathrm{nc})} \\ \sigma_{22}^{(\mathrm{nc})} \end{pmatrix} \mathrm{d}x_1, \tag{1.6}$$

$$\begin{pmatrix} A_{\mathrm{II}} \\ A_{\mathrm{I}} \end{pmatrix} = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} h(x_1) \frac{\partial}{\partial x_1} \begin{pmatrix} \sigma_{12}^{(\mathrm{nc})} \\ \sigma_{22}^{(\mathrm{nc})} \end{pmatrix} \mathrm{d}x_1. \tag{1.7}$$

Introducing instead the crack S_ε requires that

$$-\sigma_{i2}(x_1+\varepsilon\varphi,\varepsilon\psi\pm 0) \simeq \sigma_{i2}^{(\mathrm{nc})}(x_1,0)+\varepsilon\varphi\frac{\partial\sigma_{i2}^{(\mathrm{nc})}}{\partial x_1}(x_1,0)+\varepsilon\psi\frac{\partial\sigma_{i2}^{(\mathrm{nc})}}{\partial x_2}(x_1,0)$$

$$=\sigma_{i2}^{(\mathrm{nc})}(x_1,0)+\varepsilon\varphi\frac{\partial\sigma_{i2}^{(\mathrm{nc})}}{\partial x_1}(x_1,0)-\varepsilon\psi\frac{\partial\sigma_{i1}^{(\mathrm{nc})}}{\partial x_1}(x_1,0),\quad i=1,2. \tag{1.8}$$

Using the relations (1.3) and

$$\sigma_{11}^{(\mathrm{nc})}(x_1,0)=\sigma_{11}(x_1,\pm 0)-\sigma_{22}(x_1,\pm 0), \tag{1.9}$$

(see Novozhilov, 1961) together with formulae (1.8) we obtain

$$K_{\mathrm{I}} \simeq K_{\mathrm{I}}^{(0)}+\varepsilon\sqrt{\frac{\pi}{2}}(\varphi A_{\mathrm{I}}-\psi A_{\mathrm{II}}),$$

$$K_{\mathrm{II}} \simeq K_{\mathrm{II}}^{(0)}+\varepsilon\sqrt{\frac{\pi}{2}}(\varphi A_{\mathrm{II}}-\psi A_{\mathrm{I}})-\varepsilon\psi\int_{-\infty}^{0}h(x_1)\frac{\partial\sigma_{11}}{\partial x_1}\,\mathrm{d}x_1, \tag{1.10}$$

where $K_{\mathrm{I}}^{(0)}$, $K_{\mathrm{II}}^{(0)}$ represent the stress intensity factors for the unperturbed crack. This representation indicates the importance of high order terms of the asymptotic expansion of the stress field near the crack tip. Further in the text (Section 2) we discuss the case of a small perturbation of an arbitrary shape.

1.2. *Rotation of a semi-infinite crack*

Consider a rotation of the semi-infinite crack S_0 through a small angle $\varepsilon\omega$ in such a way that the resulting crack is given by

$$S_\varepsilon=\{\mathbf{x}:x_2=\tan(\varepsilon\omega)x_1\simeq\varepsilon\omega x_1, x_1<0\}. \tag{1.11}$$

Also, introduce the system of coordinates $Ox_1'x_2'$ with basis vectors

$$\begin{pmatrix}\mathbf{e}_1'\\ \mathbf{e}_2'\end{pmatrix}=\begin{pmatrix}1 & \varepsilon\omega\\ -\varepsilon\omega & 1\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix}, \tag{1.12}$$

as shown on Fig. 1(b).

For a homogeneous plane (no crack is introduced yet) the stress field admits the following expansions:

$$\sigma_{2'2'}^{(\mathrm{nc})}(x_1,\varepsilon\omega x_1)\simeq\sigma_{22}^{(\mathrm{nc})}(x_1,0)+\varepsilon\omega x_1\frac{\partial\sigma_{22}^{(\mathrm{nc})}}{\partial x_2}(x_1,0)-2\varepsilon\omega\sigma_{12}^{(\mathrm{nc})}(x_1,0)$$

$$=\sigma_{22}^{(\mathrm{nc})}(x_1,0)-\varepsilon\omega x_1\frac{\partial\sigma_{12}^{(\mathrm{nc})}}{\partial x_1}(x_1,0)-2\varepsilon\omega\sigma_{12}^{(\mathrm{nc})}(x_1,0), \tag{1.13}$$

$$\sigma_{1'2'}^{(\mathrm{nc})}(x_1,\varepsilon\omega x_1)\simeq\sigma_{12}^{(\mathrm{nc})}(x_1,0)+\varepsilon\omega x_1\frac{\partial\sigma_{12}^{(\mathrm{nc})}}{\partial x_2}(x_1,0)+\varepsilon\omega(\sigma_{22}^{(\mathrm{nc})}(x_1,0)-\sigma_{11}^{(\mathrm{nc})}(x_1,0))$$

$$=\sigma_{12}^{(\mathrm{nc})}(x_1,0)-\varepsilon\omega x_1\frac{\partial\sigma_{22}^{(\mathrm{nc})}}{\partial x_1}(x_1,0)-\frac{\partial}{\partial x_1}(\varepsilon\omega x_1 T(x_1)), \tag{1.14}$$

where we adopt the notation

$$T(x_1) = \sigma_{11}^{(\mathrm{nc})}(x_1, 0) - \sigma_{22}^{(\mathrm{nc})}(x_1, 0). \tag{1.15}$$

When we introduce a crack S_0 the quantity (1.15) can be regarded as a T-stress acting along the crack face, in view of the relations (1.3), (1.9).

Next, let us introduce an artificial assumption related to the applied stress field: it is assumed that components $\sigma_{ij}^{(\mathrm{nc})}$ have a bounded support. No doubt, this restriction is not appropriate for a real physical model, but it enables one to use expansions (1.14), (1.15) on a semi-infinite interval which is good for the purpose of the simple illustration presented here (a comprehensive asymptotic analysis is presented in Section 2). It is easily verified that

$$\int_{-\infty}^{0} \frac{\partial \sigma_{12}^{(\mathrm{nc})}}{\partial x_1}(x_1, 0)\omega x_1 \sqrt{\frac{2}{-\pi x_1}}\, \mathrm{d}x_1 = -\int_{-\infty}^{0} \sigma_{12}^{(\mathrm{nc})}(x_1, 0)\omega \frac{\partial}{\partial x_1}\left(\sqrt{\frac{-2x_1}{\pi}}\right)\mathrm{d}x_1$$

$$= -\frac{\omega}{2} K_{\mathrm{II}}^{(0)} \tag{1.16}$$

and

$$\int_{-\infty}^{0} \frac{\partial \sigma_{22}^{(\mathrm{nc})}}{\partial x_1}(x_1, 0)\omega x_1 \sqrt{\frac{-2}{\pi x_1}}\, \mathrm{d}x_1 = -\frac{\omega}{2} K_{\mathrm{I}}^{(0)}. \tag{1.17}$$

Consequently, the expansions (1.13), (1.14) together with eqns (1.6) applied to the effective tractions on S_ε, yield

$$K_{\mathrm{I}} \simeq K_{\mathrm{I}}^{(0)} - \varepsilon \frac{3\omega}{2} K_{\mathrm{II}}^{(0)},$$

$$K_{\mathrm{II}} \simeq K_{\mathrm{II}}^{(0)} + \varepsilon \frac{\omega}{2} K_{\mathrm{I}}^{(0)} - \varepsilon\omega \int_{-\infty}^{0} h(x_1)\frac{\partial}{\partial x_1}(x_1 T(x_1))\, \mathrm{d}x_1. \tag{1.18}$$

These formulae agree with the results obtained by Cotterell and Rice (1980) when specialized to the case $T = \text{const}$ that they considered. It should be mentioned that the local system of coordinates has its centre at the crack tip. It will be shown in Section 2 that an additional term in the asymptotic expansion of the stress intensity factors is required for the case when the crack tip is moved away from the origin.

It is tempting to assume that the local distribution of the stress near the crack end determines entirely the first order perturbation of the stress intensity factor for the case when the crack slightly increases its length. However, this assumption fails. It is shown explicitly in the next subsection.

1.3. *Comparison of perturbation problems for extension of finite and semi-infinite cracks*

The first example deals with a semi-infinite crack S_0 subjected to loading by a pair of forces applied on the crack faces; the second concerns a finite crack

$$S_{0,l} = \{\mathbf{x} : x_2 = 0, -l < x_1 < 0\}$$

in a plane where a remote uniform load is applied at infinity. The main objective is to evaluate the perturbation of the stress-intensity factor due to the relocation of the crack tip along the x_1-axis.

1.3.1. *A semi-infinite Mode-I crack.* Let d denote the distance between the origin and the point characterising the location of concentrated forces of intensity P. Also, suppose that the stress field satisfies the homogeneous equilibrium equations and vanishes at infinity. Then, ahead of the crack

$$\sigma_{22} = \frac{K_{\mathrm{I}}^{(0)}}{\sqrt{2\pi x_1}} \frac{d}{x_1+d} = \frac{K_{\mathrm{I}}^{(0)}}{\sqrt{2\pi x_1}} + A_1\sqrt{x_1} + \cdots, \tag{1.19}$$

where

$$K_{\mathrm{I}}^{(0)} = P\sqrt{\frac{2}{\pi d}}, \quad A_1 = -\frac{K_{\mathrm{I}}^{(0)}}{\sqrt{2\pi}}\frac{1}{d}.$$

Now consider a small increment of the crack length, so that the right end of the crack will be located at $x_1 = \varepsilon$ (the left end is supposed to be fixed). In this case the new stress intensity factor $K_{\mathrm{I}}^{(\varepsilon)}$ is given by

$$K_{\mathrm{I}}^{(\varepsilon)} = P\sqrt{\frac{2}{\pi(d+\varepsilon)}} = K_{\mathrm{I}}^{(0)} + \varepsilon c_1 A_1 + \cdots, \tag{1.20}$$

with $c_1 = \sqrt{\pi/2}$. This is consistent with the static limit of the formulae derived by Willis and Movchan (1995).

1.3.2. *A finite Mode-I crack.* Assume that the elastic plane with the crack $S_{0,l}$ is subjected to a uniform remote stress $\sigma_{22}^{\infty} = \sigma$; the body force density is supposed to be zero, and the crack faces are free of tractions. Then the stress intensity factor is given by

$$K_{\mathrm{I}}^{(0)} = \sigma\sqrt{\frac{\pi l}{2}},$$

and the asymptotic approximation of the stress σ_{22} ahead of the crack is specified by

$$\sigma_{22} = \frac{\sigma(x_1+l/2)}{\sqrt{x_1(x_1+l)}} \sim \frac{K_{\mathrm{I}}^{(0)}}{\sqrt{2\pi x_1}} + A_{1,l}\sqrt{x_1}, \tag{1.21}$$

with

$$A_{1,l} = \frac{3}{2l}\frac{K_{\mathrm{I}}^{(0)}}{\sqrt{2\pi}}.$$

Let the crack length be increased by a small amount ε in such a way that the left end of the crack does not change its position whereas the right end is relocated to the point $x_1 = \varepsilon$. The new stress-intensity factor is given by

$$\begin{aligned} K_{\mathrm{I}}^{(\varepsilon)} &= \sigma\sqrt{\frac{\pi(l+\varepsilon)}{2}} = K_{\mathrm{I}}^{(0)} + \varepsilon\frac{K_{\mathrm{I}}^{(0)}}{2l} + O(\varepsilon^2) \\ &= K_{\mathrm{I}}^{(0)} + \varepsilon c_2 A_l + O(\varepsilon^2), \end{aligned} \tag{1.22}$$

where

$$c_2 = \frac{2}{3}\sqrt{\frac{\pi}{2}} = \frac{2}{3}c_1.$$

One can observe that the constant coefficients c_1 and c_2 differ by the factor 2/3. This fact indicates that, in general, it is not enough to know the distribution of stress near the end of the unperturbed crack. The additional information required is the asymptotics of the weight functions which depend, of course, on the geometry of the entire region. We illustrate this statement through the following elementary consideration.

It can be easily verified that if ahead of the crack, located on the x_1-axis, the stress field produces

$$\sigma_{22} \sim \frac{K_{\mathrm{I}}^{(0)}}{\sqrt{2\pi x_1}} + A_1\sqrt{x_1}, \quad x_1 > 0, \tag{1.23}$$

and if the Mode-I weight function $h(x_1)$ has the asymptotic expansion

$$h(x_1) \sim \sqrt{\frac{2}{-\pi x_1}} + q\sqrt{-x_1}, \quad x_1 < 0, \tag{1.24}$$

then for a small increment ε of the crack length (we are looking at the cases where the crack is semi-infinite or its left end is fixed) the stress intensity factor at the right end is given by

$$K_{\mathrm{I}}^{(\varepsilon)} = K_{\mathrm{I}}^{(0)} + \varepsilon\sqrt{\frac{\pi}{2}}A_1 + \frac{\varepsilon}{2}\sqrt{\frac{\pi}{2}}K_{\mathrm{I}}^{(0)}q. \tag{1.25}$$

For the case of a semi-infinite crack

$$q = 0,$$

and for a finite crack of the length l

$$q = -\frac{1}{2l}\sqrt{\frac{2}{\pi}}$$

(it follows, for example, from the explicit solution for a finite crack presented in Sih and Liebowitz, 1968). Thus, for a semi-infinite crack we obtain the formula (1.20), and for the case of a finite Mode-I crack one has the result which agrees with (1.22).

In the text below we present a comprehensive asymptotic analysis of the stress-intensity factors for problems involving small perturbations of the crack front.

2. TWO-DIMENSIONAL SEMI-INFINITE CRACKS

Let

$$S_\varepsilon = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \varepsilon f(x_1), x_1 < 0\}, \tag{2.1}$$

where ε is a small positive non-dimensional parameter, and $f(x_1)$ is a smooth bounded function which tends to zero as $x_1 \to -\infty$ (see Fig. 2). Also, let $\Omega_\varepsilon = \mathbb{R}^2 \backslash S_\varepsilon$.

A crack which occupies S_ε perturbs a stress field which, in the absence of the crack, would be $\sigma_{ij}^{(\mathrm{nc})}$, with corresponding displacement field $\mathbf{u}^{(\mathrm{nc})}$, taken to be a field of plane strain. The additional displacement induced by the presence of the crack is denoted by $\mathbf{u}$. It satisfies the equilibrium equation

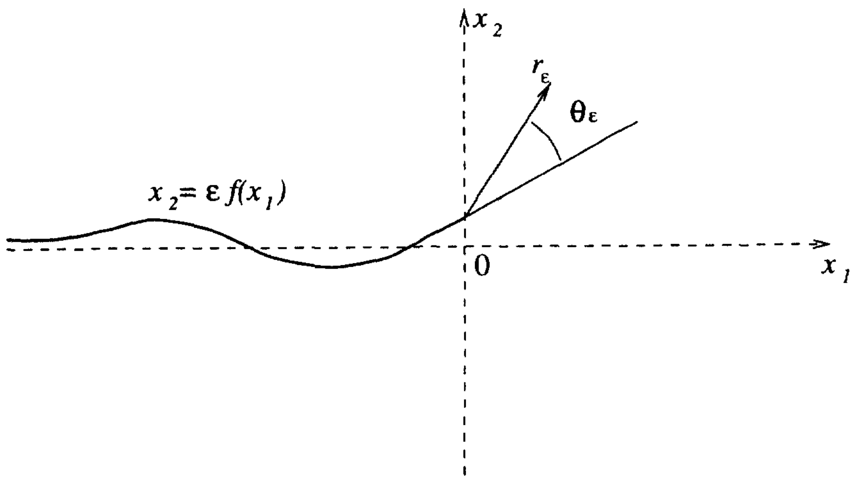

Fig. 2. An out-of-plane perturbation of a 2-D semi-infinite crack.

$$\mathbf{L}\mathbf{u} := \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega_\varepsilon, \tag{2.2}$$

with the boundary conditions

$$\sigma_i^{(n)} := \sigma_{ij}(\mathbf{u};\mathbf{x})n_j = -\sigma_{ij}^{(\mathrm{nc})}(\mathbf{x})n_j \tag{2.3}$$

on either side of $S_\varepsilon(\mathbf{x} \in S_\varepsilon^{\pm})$, and the condition

$$\mathbf{u}(\mathbf{x}) \to 0 \quad \text{as } \|\mathbf{x}\| \to \infty. \tag{2.4}$$

When $\varepsilon = 0$, the corresponding solution $\mathbf{u}^{(0)}$, defined over the domain $\Omega_0 := \mathbb{R}^2 \backslash S_0$ can be found by elementary means. In particular, the traction components $\sigma_{12}^{(0)}$, $\sigma_{22}^{(0)}$ just ahead of the crack tip, at $\mathbf{x} = (r, 0)$, are given as

$$\begin{pmatrix}\sigma_{12}^{(0)}\\ \sigma_{22}^{(0)}\end{pmatrix} \simeq \frac{1}{\sqrt{2\pi r}}\begin{pmatrix}K_{\mathrm{II}}^{(0)}\\ K_{\mathrm{I}}^{(0)}\end{pmatrix} + \sqrt{r}\begin{pmatrix}A_{\mathrm{II}}^{(0)}\\ A_{\mathrm{I}}^{(0)}\end{pmatrix},$$

where [compare with (1.5)]

$$\begin{pmatrix}K_{\mathrm{II}}^{(0)}\\ K_{\mathrm{I}}^{(0)}\end{pmatrix} = \int_{-\infty}^{0} h(x_1)\begin{pmatrix}\sigma_{12}^{(\mathrm{nc})}\\ \sigma_{22}^{(\mathrm{nc})}\end{pmatrix}\mathrm{d}x_1,$$

$$\begin{pmatrix}A_{\mathrm{II}}^{(0)}\\ A_{\mathrm{I}}^{(0)}\end{pmatrix} = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{0} h(x_1)\frac{\partial}{\partial x_1}\begin{pmatrix}\sigma_{12}^{(\mathrm{nc})}\\ \sigma_{22}^{(\mathrm{nc})}\end{pmatrix}\mathrm{d}x_1.$$

We consider separately two cases: (i) when the unperturbed crack is subjected to Mode-I loading ($\sigma_{12}^{(\mathrm{nc})}(x_1, 0) = 0$); (ii) when the unperturbed crack is subjected to Mode-II loading ($\sigma_{22}^{(\mathrm{nc})}(x_1, 0) = 0$).

2.1. *Perturbation of the Mode-I crack*

In the vicinity of the crack tip, the displacement field $\mathbf{u}$ admits the following asymptotic representation, relative to local polar coordinates $(r_\varepsilon, \theta_\varepsilon)$ as illustrated in Fig. 2:

$$\mathbf{u} \sim \sum_{j=\mathrm{I,II}} \{r_\varepsilon^{1/2} K_j(\varepsilon)\mathbf{\Phi}^{(j)}(\theta_\varepsilon) + r_\varepsilon^{3/2}\sqrt{2\pi}A_j(\varepsilon)\mathbf{\Xi}^{(j)}(\theta_\varepsilon)\} + \mathbf{c}(\varepsilon) + T(\varepsilon)r_\varepsilon\boldsymbol{\chi}(\theta_\varepsilon) \quad \text{as } r_\varepsilon \to 0. \tag{2.5}$$

The coefficients K_j are the stress intensity factors, and A_j provide the coefficients of the "next terms" in the traction components ahead of the crack; $\mathbf{c}(\varepsilon)$ is a constant vector and $T(\varepsilon)$ is the T-stress. The angular functions $\mathbf{\Phi}^{(j)}$, $\mathbf{\Xi}^{(j)}$, $\boldsymbol{\chi}$ are given by (see Williams, 1957)

$$\mathbf{\Phi}^{(\mathrm{I})}(\theta) = \begin{pmatrix}\Phi_r^{(\mathrm{I})}(\theta)\\ \Phi_\theta^{(\mathrm{I})}(\theta)\end{pmatrix} = \frac{1}{4\mu\sqrt{2\pi}}\begin{bmatrix}(2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\\ -(2\kappa+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\end{bmatrix}, \tag{2.6}$$

$$\mathbf{\Phi}^{(\mathrm{II})}(\theta) = \begin{pmatrix}\Phi_r^{(\mathrm{II})}(\theta)\\ \Phi_\theta^{(\mathrm{II})}(\theta)\end{pmatrix} = \frac{1}{4\mu\sqrt{2\pi}}\begin{bmatrix}-(2\kappa-1)\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2}\\ -(2\kappa+1)\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\end{bmatrix}, \tag{2.7}$$

$$\mathbf{\Xi}^{(\mathrm{I})}(\theta) = \begin{pmatrix}\Xi_r^{(\mathrm{I})}(\theta)\\ \Xi_\theta^{(\mathrm{I})}(\theta)\end{pmatrix} = \frac{1}{12\mu\sqrt{2\pi}}\begin{bmatrix}(2\kappa-3)\cos\frac{\theta}{2} + \cos\frac{5\theta}{2}\\ (2\kappa+3)\sin\frac{\theta}{2} - \sin\frac{5\theta}{2}\end{bmatrix}, \tag{2.8}$$

$$\mathbf{\Xi}^{(\mathrm{II})}(\theta) = \begin{pmatrix}\Xi_r^{(\mathrm{II})}(\theta)\\ \Xi_\theta^{(\mathrm{II})}(\theta)\end{pmatrix} = \frac{1}{12\mu\sqrt{2\pi}}\begin{bmatrix}(2\kappa-3)\sin\frac{\theta}{2} + 5\sin\frac{5\theta}{2}\\ -(2\kappa+3)\cos\frac{\theta}{2} + 5\cos\frac{5\theta}{2}\end{bmatrix}, \tag{2.9}$$

$$\boldsymbol{\chi}(\theta) = \begin{pmatrix}\chi_r\\ \chi_\theta\end{pmatrix} = \frac{1}{8\mu}\begin{pmatrix}2\cos(2\theta) + \kappa - 1\\ -2\sin(2\theta)\end{pmatrix}, \tag{2.10}$$

where $\kappa = (\lambda+3\mu)/(\lambda+\mu) = 3-4\nu$; ν is the Poisson ratio.

The stress intensity factors are assumed to depend smoothly on the small parameter ε. Thus,

$$\begin{pmatrix}K_\mathrm{I}(\varepsilon)\\ K_\mathrm{II}(\varepsilon)\end{pmatrix} \sim \begin{pmatrix}K_\mathrm{I}(0)\\ 0\end{pmatrix} + \varepsilon\begin{pmatrix}K'_\mathrm{I}\\ K'_\mathrm{II}(0)\end{pmatrix} + O(\varepsilon^2).$$

Here, $K_\mathrm{II}(0) = 0$ because the unperturbed crack S_0 is subject to Mode-I loading. We employ the notations

$$K_j^{(0)} = K_j(0), \quad j = \mathrm{I, II}.$$

Further analysis will provide the derivatives $K'_j(0)$, and hence the first order corrections to the stress intensity factors.

A formal solution of the problem (2.2)–(2.4) may be developed as an asymptotic power series

$$\mathbf{u}(\mathbf{x},\varepsilon) \sim \mathbf{u}^{(0)}(\mathbf{x})+\varepsilon\mathbf{u}^{(1)}(\mathbf{x})+\cdots \tag{2.11}$$

with respect to ε. The principal part $\mathbf{u}^{(0)}$ satisfies the boundary value problem for the limit region $\boldsymbol{\Omega}_0$

$$\mathbf{L}\mathbf{u}^{(0)}(\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega_0, \tag{2.12}$$

$$\sigma_{i2}(\mathbf{u}^{(0)}, x_1, \pm 0) = -\sigma_{i2}^{(\mathrm{nc})}(x_1, 0), \quad -\infty < x_1 < 0, \tag{2.13}$$

$$\mathbf{u}^{(0)}(\mathbf{x}) \to 0, \quad \text{as } \|\mathbf{x}\| \to \infty. \tag{2.14}$$

Formally, for the second term of the asymptotic series (2.11) one can write

$$\mathbf{L}\mathbf{u}^{(1)}(\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega_0, \tag{2.15}$$

$$\sigma_{2j}(\mathbf{u}^{(1)}; x_1, \pm 0) = \frac{\partial}{\partial x_1}\{f(x_1)\sigma_{1j}(\mathbf{u}^{(0)}; x_1, \pm 0)\}, \quad -\infty < x_1 < 0, \tag{2.16}$$

$$\mathbf{u}^{(1)} \to 0 \quad \text{as } \|\mathbf{x}\| \to \infty. \tag{2.17}$$

Also, it is assumed that $\mathbf{u}^{(1)}$ vanishes at infinity.

Strictly, the series (2.11) is an outer expansion, not valid in a boundary layer distant $O(\varepsilon)$ from S_0. The boundary conditions (2.16) define its second term, because the tractions specified there are bounded, and equilibrium can be imposed uniformly across the boundary layer.

Using (2.13) and (2.16) we can state more precisely that

$$\sigma_{22}(\mathbf{u}^{(1)}; x_1, \pm 0) = 0, \quad \sigma_{21}(\mathbf{u}^{(1)}; x_1, \pm 0) = \frac{\partial}{\partial x_1}\{f(x_1)\sigma_{11}(\mathbf{u}^{(0)}; x_1, \pm 0)\}. \tag{2.18}$$

Let (r, θ) denote polar coordinates related to the tip of the reference crack S_0. Then, the field $\mathbf{u}^{(0)}$ admits the asymptotic approximation

$$\mathbf{u}^{(0)} \sim \mathbf{c}(0)+r^{1/2}\{K_{\mathrm{I}}^{(0)}\boldsymbol{\Phi}^{(\mathrm{I})}(\theta)+K_{\mathrm{II}}^{(0)}\boldsymbol{\Phi}^{(\mathrm{II})}(\theta)\} \quad \text{as } r \to 0. \tag{2.19}$$

(In our particular case $K_{\mathrm{II}}^{(0)} = 0$.) Direct calculation gives

$$\sigma_{11}(\mathbf{c}(0)+r^{1/2}\{K_{\mathrm{I}}^{(0)}\boldsymbol{\Phi}^{(\mathrm{I})}(\theta)+K_{\mathrm{II}}^{(0)}\boldsymbol{\Phi}^{(\mathrm{II})}(\theta)\})\,|_{\theta=\pm\pi} = \mp\left(\frac{2}{\pi r}\right)^{1/2} K_{\mathrm{II}}^{(0)}. \tag{2.20}$$

Then, it follows from (2.18), (2.20) that

$$\sigma_{21}(\mathbf{u}^{(1)}; x_1, \pm 0) = O(1).$$

It should be emphasized that in the general case involving Mode-II loading, singular terms occur in the formally derived traction boundary conditions for $\mathbf{u}^{(1)}$. As we show below these terms indicate the presence of the boundary layer which occurs near the crack tip due to relocation of the crack front. For the case of Mode-I loading we are just lucky to have bounded tractions in the problem for $\mathbf{u}^{(1)}$.

Let us introduce local coordinates $\mathbf{y} = (y_1, y_2)$ corresponding to the perturbed crack S_ε:

$$\mathbf{y} = \begin{pmatrix} 1 & \varepsilon\omega \\ -\varepsilon\omega & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 0 \\ \varepsilon\psi \end{pmatrix}, \tag{2.21}$$

where $\psi = f(0)$ and to first order approximation $\omega = f'(0)$. The system of coordinates $\mathbf{Y}^{(\varepsilon)}$ has its origin at the end of the crack S_ε, and the orientation of the axes corresponds to the orientation of the crack contour at $x_1 = 0$.

The following expansion holds

$$\begin{aligned}\mathbf{u} &= \mathbf{u}^{(0)} + \varepsilon \frac{\partial}{\partial \varepsilon}\left\{\begin{pmatrix} 1 & -\varepsilon\omega \\ \varepsilon\omega & 1 \end{pmatrix}\mathbf{u}(\mathbf{y})\right\}\Bigg|_{\varepsilon=0} + O(\varepsilon^2) \\ &= \mathbf{u}^{(0)} + \varepsilon\mathbf{c}'(0) + \varepsilon\left\{\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\mathbf{u}^{(0)}(\mathbf{y}) + \frac{\partial}{\partial\varepsilon}(\mathbf{u}(\mathbf{y}))\Bigg|_{\varepsilon=0}\right\} + O(\varepsilon^2).\end{aligned} \tag{2.22}$$

It can be verified by direct calculation that the representation (2.22) is equivalent to

$$\begin{aligned}\mathbf{u} = \mathbf{u}^{(0)} + \varepsilon\mathbf{c}'(0) + \varepsilon \sum_{j=\mathrm{I,II}} \Bigg\{ & K_j'(0) r^{1/2}\mathbf{\Phi}^{(j)}(\theta) + K_j(0)\Bigg[-\psi\frac{\partial}{\partial y_2}(r^{1/2}\mathbf{\Phi}^{(j)}(\theta)) \\ & + \omega\left(y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}\right)(r^{1/2}\mathbf{\Phi}^{(j)}(\theta)) + \omega r^{1/2}(-\Phi_2^{(j)}(\theta), \Phi_1^{(j)}(\theta))'\Bigg] \\ & - \psi\sqrt{2\pi}A_j(0)\frac{\partial}{\partial y_2}(r^{3/2}\mathbf{\Xi}^{(j)}(\theta))\Bigg\} + \text{smaller terms.}\end{aligned} \tag{2.23}$$

We shall also need the following vector functions which satisfy a homogeneous Lamé system and homogeneous traction boundary conditions on the faces of the crack S_0.

$$\zeta^{(j)}(\mathbf{x}) = r^{-1/2}\mathbf{\Psi}^{(j)}(\theta), \quad j = \mathrm{I, II} \tag{2.24}$$

where

$$\mathbf{\Psi}^{(\mathrm{I})} = \begin{pmatrix} \Psi_r^{(\mathrm{I})} \\ \Psi_\theta^{(\mathrm{I})} \end{pmatrix} = -\frac{1}{(1+\kappa)\sqrt{8\pi}}\left[\begin{array}{c} (2\kappa+1)\cos\frac{3\theta}{2} - 3\cos\frac{\theta}{2} \\ -(2\kappa-1)\sin\frac{3\theta}{2} + 3\sin\frac{\theta}{2} \end{array}\right],$$

$$\mathbf{\Psi}^{(\mathrm{II})} = \begin{pmatrix} \Psi_r^{(\mathrm{II})} \\ \Psi_\theta^{(\mathrm{II})} \end{pmatrix} = -\frac{1}{(1+\kappa)\sqrt{8\pi}}\left[\begin{array}{c} -(2\kappa+1)\sin\frac{3\theta}{2} + \sin\frac{\theta}{2} \\ -(2\kappa-1)\cos\frac{3\theta}{2} + \cos\frac{\theta}{2} \end{array}\right].$$

The following relations are useful

$$\frac{\partial}{\partial y_2}(r^{1/2}\mathbf{\Phi}^{(\mathrm{I})}(\theta)) = \frac{1+\kappa}{4\mu}\zeta^{(\mathrm{II})}(\mathbf{x}), \tag{2.25}$$

$$\left(y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}\right)(r^{1/2}\mathbf{\Phi}^{(\mathrm{I})}(\theta)) + r^{1/2}(\Phi_1^{(\mathrm{I})}(\theta)\mathbf{e}_2 - \Phi_2^{(\mathrm{I})}(\theta)\mathbf{e}_1) = -\frac{1}{2}r^{1/2}\mathbf{\Phi}^{(\mathrm{II})}(\theta), \tag{2.26}$$

and

$$\frac{\partial}{\partial y_2}(r^{3/2}\boldsymbol{\Xi}^{(j)}(\theta)) = -\frac{1}{2}r^{1/2}\boldsymbol{\Phi}^{(\mathrm{II})}(\theta). \tag{2.27}$$

It follows from (2.23) and (2.25)–(2.27) that (here the assumption $K_{\mathrm{II}}(0) = A_{\mathrm{II}}(0) = 0$ is used)

$$\mathbf{u}^{(1)} = \mathbf{c}'(0) - \psi\frac{1+\kappa}{4\mu}K_{\mathrm{I}}(0)\boldsymbol{\zeta}^{(\mathrm{II})}(\mathbf{x}) + r^{1/2}\left\{K'_{\mathrm{I}}(0)\boldsymbol{\Phi}^{(\mathrm{I})}(\theta) + \left(\sqrt{\frac{\pi}{2}}\psi A_{\mathrm{I}}(0) - \frac{1}{2}\omega K_{\mathrm{I}}(0) + K'_{\mathrm{II}}(0)\right)\boldsymbol{\Phi}^{(\mathrm{II})}(\theta)\right\} + \text{smaller terms}. \tag{2.28}$$

We emphasize that the coefficients $K_{\mathrm{I}}(0)$ and $A_{\mathrm{I}}(0)$, corresponding to the unperturbed crack, are given.

Clearly, the second term in (2.28) is characterized by a high singularity (the corresponding energy integral is infinite). Formally, this singular term occurs due to relocation of the coordinate system from the actual crack tip to the end of the reference crack S_0. Physically, it indicates existence of a boundary layer in a neighbourhood of the perturbed crack front; the expansion (2.28) must be treated as an outer expansion which is valid in the exterior of a neighbourhood of the crack.

In this particular paper we have no intention to analyze the boundary layer. We have to obtain just the quantity $K'_{\mathrm{I}}(0)$ characterising the perturbation of the stress intensity factor.

Let us consider an auxiliary field

$$\mathbf{v}^{(1)} = \mathbf{u}^{(1)} + \frac{1+\kappa}{4\mu}\psi K_{\mathrm{I}}(0)\boldsymbol{\zeta}^{(\mathrm{II})}. \tag{2.29}$$

The vector function $\mathbf{v}^{(1)}$ satisfies the homogeneous Lamé system (2.15) and the boundary conditions (2.16). It also vanishes at infinity and does not have a singularity at the tip of the reference crack S_0; as $r \to 0$ the vector function $\mathbf{v}^{(1)}$ admits the asymptotic approximation

$$\mathbf{v}^{(1)} \sim r^{1/2}\left\{K'_{\mathrm{I}}(0)\boldsymbol{\Phi}^{(\mathrm{I})}(\theta) + \left(\sqrt{\frac{\pi}{2}}\psi A_{\mathrm{I}}(0) - \frac{1}{2}\omega K_{\mathrm{I}}(0) + K'_{\mathrm{II}}(0)\right)\boldsymbol{\Phi}^{(\mathrm{II})}(\theta)\right\}. \tag{2.30}$$

Using standard technique we obtain (the details of the calculations are presented in Appendix A)

$$K'_{\mathrm{II}}(0) - \frac{1}{2}\omega K_{\mathrm{I}}(0) + \sqrt{\frac{\pi}{2}}\psi A_{\mathrm{I}}(0) - D = 0, \tag{2.31}$$

where

$$D = -\int_{-\infty}^{0} h(x_1)\frac{\partial}{\partial x_1}\left\{f(x_1)\sigma_{11}(\mathbf{u}^{(0)}; x_1, +0)\right\}\mathrm{d}x_1,$$

with h being the weight function (1.4).

Consequently, the stress intensity factor K_{II} is approximated by

$$K_{II}(\varepsilon) = \varepsilon K'_{II}(0) + o(\varepsilon) = \varepsilon\left(\frac{1}{2}\omega K_I(0) - \sqrt{\frac{\pi}{2}}\psi A_I(0) + D\right) + o(\varepsilon). \qquad (2.32)$$

One can also show that $K'_I(0) = 0$. Hence,

$$K_I(\varepsilon) = K_I(0) + o(\varepsilon). \qquad (2.33)$$

In the next subsection we consider the shear mode crack and asymptotic formulae for the stress-intensity factors.

2.2. *The Mode-II crack*

Now suppose that the displacement vector $\mathbf{u}(\mathbf{x})$ satisfies the homogeneous system (2.2) and the condition (2.4) at infinity, but

$$\sigma_{12}^{(nc)} \neq 0, \quad \sigma_{22}^{(nc)} = 0. \qquad (2.34)$$

Again, we use the asymptotic formula (2.11). In our particular case the coefficient $\mathbf{u}^{(1)}$ satisfies the system (2.15), it vanishes at infinity, and the formal boundary conditions (2.18) are replaced by

$$\sigma_{22}(\mathbf{u}^{(1)}; x_1, \pm 0) = -\frac{\partial}{\partial x_1}\{\sigma_{12}^{(nc)}(x_1, 0) f(x_1)\},$$

$$\sigma_{21}(\mathbf{u}^{(1)}; x_1, \pm 0) = \frac{\partial}{\partial x_1}\{f(x_1)\sigma_{11}(\mathbf{u}^{(0)}; x_1, \pm 0)\}. \qquad (2.35)$$

The asymptotic formula (2.5) remains valid near the actual crack tip. In the present case,

$$K_I(0) = A_I(0) = 0,$$

and it follows from (2.23) (which is written for the general case) that

$$\mathbf{u}^{(1)} = -\psi K_{II}(0)\left\{-\frac{1+\kappa}{4\mu}r^{-1/2}\mathbf{\Psi}^{(I)}(\theta) + r^{-1/2}\mathbf{\Upsilon}^{(I)}(\theta)\right\} + K'_{II}(0)r^{1/2}\mathbf{\Phi}^{(II)}(\theta)$$

$$+\left(-\sqrt{\frac{\pi}{2}}\psi A_{II}(0) + \frac{1}{2}\omega K_{II}(0) + K'_I(0)\right)r^{1/2}\mathbf{\Phi}^{(I)}(\theta)$$

$$-(\sqrt{2\pi}\psi A_{II}(0) + \omega K_{II}(0))r^{1/2}\mathbf{\Upsilon}^{(II)}(\theta), \qquad (2.36)$$

where the vector functions $\mathbf{\Upsilon}^{(j)}, j = \mathrm{I}, \mathrm{II}$ are given by

$$\mathbf{\Upsilon}^{(I)} = \begin{pmatrix}\Upsilon_r^{(I)} \\ \Upsilon_\theta^{(I)}\end{pmatrix} = \frac{1}{\mu\sqrt{2\pi}}\begin{bmatrix}\cos\dfrac{\theta}{2} \\ -\sin\dfrac{\theta}{2}\end{bmatrix},$$

$$\mathbf{\Upsilon}^{(II)} = \begin{pmatrix}\Upsilon_r^{(II)} \\ \Upsilon_\theta^{(II)}\end{pmatrix} = \frac{1}{\mu\sqrt{2\pi}}\begin{bmatrix}\cos\dfrac{3\theta}{2} \\ -\sin\dfrac{3\theta}{2}\end{bmatrix}. \qquad (2.37)$$

The fields $r^{-1/2}\boldsymbol{\Upsilon}^{(\mathrm{I})}$, $r^{1/2}\boldsymbol{\Upsilon}^{(\mathrm{II})}$ satisfy the homogeneous Lamé system, but the shear components of tractions do not vanish. Formally we can write

$$\sigma_{22}(r^{-1/2}\boldsymbol{\Upsilon}^{(\mathrm{I})})|_{\theta=\pm\pi} = \sigma_{22}(r^{1/2}\boldsymbol{\Upsilon}^{(\mathrm{II})})|_{\theta=\pm\pi} = 0,$$

$$\sigma_{12}(r^{-1/2}\boldsymbol{\Upsilon}^{(\mathrm{I})})|_{\theta=\pm\pi} = \pm\frac{1}{\sqrt{2\pi}}r^{-3/2}, \quad \sigma_{12}(r^{1/2}\boldsymbol{\Upsilon}^{(\mathrm{II})})|_{\theta=\pm\pi} = \pm\frac{1}{\sqrt{2\pi}}r^{-1/2}. \qquad (2.38)$$

As before, the field $\mathbf{u}^{(1)}$ has a strong singularity which indicates the presence of a boundary layer in a neighbourhood of the actual perturbed crack. So, practically, one has to regard (2.36) as the term in an outer expansion corresponding to the first correction term in the representation of the displacement field.

We shall try to use a trick which is similar to one employed in Section 2.1. Namely, we introduce an auxiliary vector function

$$\mathbf{U} = \psi K_{\mathrm{II}}(0)\left\{\frac{1+\kappa}{4\mu}r^{-1/2}\boldsymbol{\Psi}^{(\mathrm{I})}(\theta) - r^{-1/2}\boldsymbol{\Upsilon}^{(\mathrm{I})}(\theta)\right\} - (\sqrt{2\pi}\psi A_{\mathrm{I}}(0)+\omega K_{\mathrm{II}}(0))r^{1/2}\boldsymbol{\Upsilon}^{(\mathrm{II})}(\theta),$$

and then consider

$$\mathbf{v}^{(1)} = \mathbf{u}^{(1)} - \mathbf{U}. \qquad (2.39)$$

This vector function satisfies the homogeneous Lamé system

$$\mathbf{L}\mathbf{v}^{(1)} = 0, \quad \text{in } \mathbb{R}^2\backslash \mathrm{S}_0, \qquad (2.40)$$

and the following traction boundary conditions

$$\sigma_{22}(\mathbf{v}^{(1)}; x_1, \pm 0) = -\frac{\partial}{\partial x_1}\{f(x_1)\sigma_{12}^{(\mathrm{nc})}(x_1, 0)\}, \qquad (2.41)$$

$$\sigma_{12}(\mathbf{v}^{(1)}; x_1, \pm 0) = \frac{\partial}{\partial x_1}\{f(x_1)\sigma_{11}(\mathbf{u}^{(0)}; x_1, \pm 0)\} \pm \psi\frac{K_{\mathrm{II}}(0)}{(2\pi r^3)^{1/2}}$$
$$\pm(\sqrt{2\pi}\psi A_{\mathrm{II}}(0)+\omega K_{\mathrm{II}}(0))\frac{1}{\sqrt{2\pi r}}. \qquad (2.42)$$

Note that, as we approach the crack tip, the first term on the right-hand side (2.42) is singular, and the second and third terms compensate this singularity, so that, as a result, the right-hand side of (2.42) is bounded in a neighbourhood of the crack end. In contrast with the case related to the Mode-I crack, the vector function $\mathbf{v}^{(1)}$ does not decay at infinity: it is characterised by the following asymptotic formula

$$\mathbf{v}^{(1)} \sim (\sqrt{2\pi}\psi A_{\mathrm{II}}(0)+\omega K_{\mathrm{II}}(0))r^{1/2}\boldsymbol{\Upsilon}^{(\mathrm{II})}(\theta) \quad \text{as } r\to\infty. \qquad (2.43)$$

It follows from (2.39) that in the vicinity of the crack end

$$\mathbf{v}^{(1)} \sim K'_{\mathrm{II}}(0)r^{1/2}\boldsymbol{\Phi}^{(\mathrm{II})}(\theta) + \left(\frac{1}{2}\omega K_{\mathrm{II}}(0) - \sqrt{\frac{\pi}{2}}\psi A_{\mathrm{II}}(0) + K'_{\mathrm{I}}(0)\right)r^{1/2}\boldsymbol{\Phi}^{(\mathrm{I})}(\theta), \quad \text{as } r\to 0.$$

Now, in order to evaluate $K'_{\mathrm{I}}(0)$ and $K'_{\mathrm{II}}(0)$ we apply the Betti formula in the region $B_R\backslash S_0$ to the vector functions $\mathbf{v}^{(1)}$, $\boldsymbol{\zeta}^{(\mathrm{I})}$ and $\mathbf{v}^{(1)}$, $\boldsymbol{\zeta}^{(\mathrm{II})}$, where B_R is the ball $\{\mathbf{x} : \|\mathbf{x}\| < R\}$. The detailed calculations are presented in Appendix A. Taking the limit $R\to\infty$ we obtain

$$K_{\mathrm{I}}'(0) = -\frac{3}{2}\omega K_{\mathrm{II}}(0) - \sqrt{\frac{\pi}{2}}\psi A_{\mathrm{II}}(0) \tag{2.44}$$

and

$$K_{\mathrm{II}}'(0) = 0. \tag{2.45}$$

Consequently, for the case when the unperturbed crack corresponds to the Mode-II state,

$$K_{\mathrm{I}}(\varepsilon) = -\varepsilon\left(\frac{3}{2}\omega K_{\mathrm{II}}^{(0)} + \sqrt{\frac{\pi}{2}}\psi A_{\mathrm{II}}(0)\right) + o(\varepsilon), \tag{2.46}$$

and

$$K_{\mathrm{II}}(\varepsilon) = K_{\mathrm{II}}^{(0)} + o(\varepsilon). \tag{2.47}$$

Clearly, for general loading a combination of formulae (2.32), (2.33) and (2.46), (2.47) can be used. (We deliberately assume that the load is applied outside of the perturbation area, otherwise one would need to deal with series expansions of applied tractions with respect ε; it does not produce any difficulties, but just yields some additional terms in the traction boundary conditions involved in the problem for $\mathbf{u}^{(1)}$.)

2.3. *Comparison with formulae of Cotterell and Rice*

The problem described in subsections 2.1 and 2.2 is not new, We refer to the classical paper of Cotterell and Rice (1980) where the perturbation of the stress intensity factors was analyzed for the case of a small deviation of the crack trajectory. These authors introduced some additional restrictions: namely the T-stress was supposed to be constant and also the local system of coordinates was relocated to the tip of the actual perturbed crack (to avoid apparent singularities in the asymptotics of the displacement field which occur due to a singular perturbation of the boundary). Cotterell and Rice (1980) derived the following approximation

$$K_{\mathrm{I}} \simeq \hat{K}_{\mathrm{I}}^{(0)} - \varepsilon\tfrac{3}{2}\omega K_{\mathrm{II}}^{(0)}, \tag{2.48}$$

$$K_{\mathrm{II}} \simeq \hat{K}_{\mathrm{II}}^{(0)} + \varepsilon\left(\frac{\omega}{2}K_{\mathrm{I}}^{(0)} - \sigma_{11}^{(0)}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}\frac{f'(x_1)}{\sqrt{-x_1}}\mathrm{d}x_1\right). \tag{2.49}$$

(For the sake of convenience we have adopted the notations used in the first two parts of the section.) One should emphasize that the coefficients $\hat{K}_j^{(0)}, j = \mathrm{I}, \mathrm{II}$ in (2.48), (2.49) differ from $K_j^{(0)}$ in (2.32), (2.46), because of the relocation of the local system of coordinates to the end of the perturbed crack.

In the general case when both the longitudinal and transverse perturbations occur, so that

$$S_\varepsilon = \{\mathbf{x} : x_2 = \varepsilon f(x_1), x_1 < \varepsilon\varphi\},$$

the asymptotic formulae for the stress-intensity factors, that follow from Section 1.1, 2.1, 2.2, have the form

$$K_{\rm I} \simeq K_{\rm I}^{(0)} - \varepsilon \left\{ \frac{3}{2}\omega K_{\rm II}^{(0)} + \sqrt{\frac{\pi}{2}}(\psi A_{\rm II}(0) - \varphi A_{\rm I}(0)) \right\}, \tag{2.50}$$

$$K_{\rm II} \simeq K_{\rm II}^{(0)} + \varepsilon \left\{ \frac{\omega}{2} K_{\rm I}^{(0)} - \sqrt{\frac{\pi}{2}}(\psi A_{\rm I}(0) - \varphi A_{\rm II}(0)) - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} \frac{1}{\sqrt{-x_1}} \frac{\partial}{\partial x_1} \{ f(x_1) \sigma_{11}(\mathbf{u}^{(0)}; x_1, +0) \} \, dx_1 \right\}. \tag{2.51}$$

As in the previous sections, we use the notations $\psi = f(0)$, $\omega = f'(0)$, and $\mathbf{u}^{(0)}$ denotes the displacement field associated with the unperturbed crack. These formulae agree with (2.48) and (2.49) in the special case that $\varphi = \psi = 0$ and $\sigma_{11}(\mathbf{u}^{(0)}; x_1, +0)$ is constant.

3. THREE-DIMENSIONAL SEMI-INFINITE CRACK. OUT-OF-PLANE DEFLECTION

3.1. *Formulation*

Here we consider a three-dimensional perturbation of a plane crack. The surface of the perturbed crack is S_ε, where

$$S_\varepsilon = \{ \mathbf{x} : x_3 = \varepsilon \psi(x_1, x_2), x_1 < 0 \}. \tag{3.1}$$

The function $\psi(x_1, x_2)$ is assumed to be smooth and bounded (see Fig. 3). The unperturbed plane crack has surface S_0, to which S_ε reduces when $\varepsilon = 0$. In-plane perturbation of the crack front was analyzed by Gao and Rice (1989). Also, the solutions of Willis and Movchan (1995) and Movchan and Willis (1995) for the dynamic in-plane perturbation of a propagating crack reproduce the results of Gao and Rice in the static limit.

The medium is linearly elastic and isotropic, and it is assumed that the crack perturbs displacement and stress fields $\mathbf{u}^{(nc)}$ and $\boldsymbol{\sigma}^{(nc)}$. These are as introduced in Section 1.1 except that now they depend upon x_1 and x_3, and an unperturbed Mode-III component is also admitted. It is assumed, however, that the body force associated with the field $\mathbf{u}^{(nc)}$ has a compact support which does not intersect the crack surface S_ε. The additional displacement introduced by the presence of the crack is denoted by $\mathbf{u}(\mathbf{x}; \varepsilon)$. It satisfies the homogeneous Lamé system

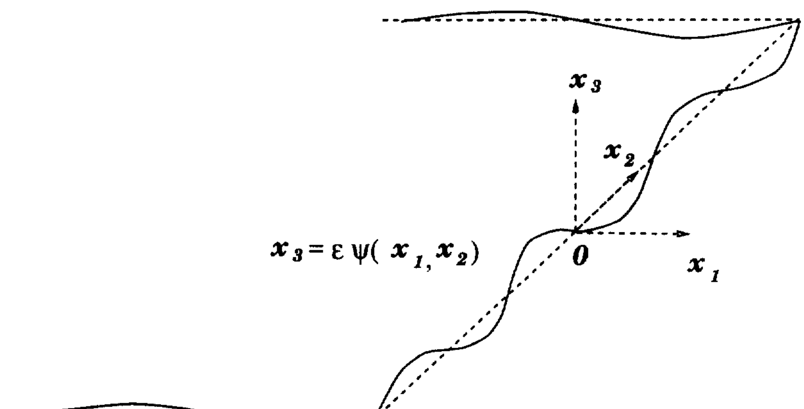

Fig. 3. A slightly nonplanar 3-D crack.

$$\mathbf{Lu}(\mathbf{x};\varepsilon) = 0 \quad \text{in } \mathbb{R}^3 \backslash \mathrm{S}_\varepsilon \tag{3.2}$$

and the traction boundary conditions on the crack surface

$$[\mathbf{n}\cdot\boldsymbol{\sigma}(\mathbf{u};\mathbf{x})+\mathbf{n}\cdot\boldsymbol{\sigma}^{(\mathrm{inc})}(\mathbf{x})]_{S_\varepsilon} = 0. \tag{3.3}$$

This displacement $\mathbf{u}(\mathbf{x};\varepsilon)$ tends to zero as $\|\mathbf{x}\| \to \infty$ and is discontinuous across S_ε.

In the particular case $\varepsilon = 0$, the displacement $\mathbf{u}(\mathbf{x};0)$ is written as $\mathbf{u}^{(0)}(\mathbf{x})$, and the corresponding stress is $\sigma^{(0)}(\mathbf{x})$. It is convenient then to employ the notation

$$\Delta\mathbf{u} = \mathbf{u}(\mathbf{x};\varepsilon)-\mathbf{u}^{(0)}(\mathbf{x}), \quad \Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u};\mathbf{x})-\boldsymbol{\sigma}^{(0)}(\mathbf{x}). \tag{3.4}$$

The objective is to find expressions for the corresponding stress intensity factors $K_j(x_2;\varepsilon)$ ($j = \mathrm{I,II,III}$) or, equivalently, for the perturbations

$$\Delta K_j = K_j - K_j^{(0)}, \tag{3.5}$$

where $K_j^{(0)} = K_j(x_2;0)$.

3.2. *The fundamental identity*

While it might be possible to develop an asymptotic algorithm along the lines presented in Section 2, it is evident that the third dimension would introduce major complications. It is, in fact, possible to proceed much more directly, by making a modest adaptation of a method introduced for in-plane perturbations by Willis and Movchan (1995) and Movchan and Willis (1995). This is outlined now.

Take $d > 0$. Then, for all ε smaller than some ε_0, the crack surface S_ε is contained entirely within the region $\{\mathbf{x}: -d < x_3 < d\}$. Now let $\mathbf{u}'(\mathbf{x})$, $\boldsymbol{\sigma}'(\mathbf{x})$ be displacement and stress fields that satisfy the homogeneous Lamé system for all $x_3 < 0$ and all $x_3 > 0$, and suppose that $\mathbf{u}'(\mathbf{x})$ decays to zero at some suitable rate (specified precisely later) as $\|\mathbf{x}\| \to \infty$. Observe too that the displacement and stress pair

$$\mathbf{u}''(\mathbf{x}) = \mathbf{u}'(-\mathbf{x}), \quad \boldsymbol{\sigma}''(\mathbf{x}) = -\boldsymbol{\sigma}'(-\mathbf{x})$$

also satisfy the conditions specified for $\mathbf{u}'$, $\boldsymbol{\sigma}'$.

Now apply Betti's reciprocal theorem to large hemispheres

$$B_\pm = \{\mathbf{x}: \pm x_3 > \pm d, \|\mathbf{x}\| < R\},$$

for the fields $\mathbf{u}(\mathbf{x};\varepsilon)$ and $\mathbf{u}''(\mathbf{x}-\mathbf{x}')$ where $\mathbf{x}'$ has components $(x_1', x_2', 0)$. With suitable assumptions of decay at infinity, taking the limit as $R \to \infty$ yields

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \sum_{i=1}^{3} (u_i''(x_1 - x_1', x_2 - x_2', \pm d)\sigma_{i3}(\mathbf{u};\varepsilon)\,|_{x_3=\pm d} - \sigma_{i3}''(x_1 - x_1', x_2 - x_2', \pm d)u_i(x_1, x_2, \pm d;\varepsilon)) = 0. \tag{3.6}$$

Taking the difference between those two identities now yields

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \left[\sum_{i=1}^{3} (u_i''(x_1 - x_1', x_2 - x_2', x_3)\sigma_{i3}(\mathbf{u}; \varepsilon) - \sigma_{i3}''(x_1 - x_1', x_2 - x_2', x_3) u_i(x_1, x_2, \pm d; \varepsilon)) \right]_{x_3 = -d}^{x_3 = d} = 0. \quad (3.7)$$

For an arbitrary function $f(x_1, x_2, x_3)$ we introduce the notations

$$[f]_d(x_1, x_2) = f(x_1, x_2, d) - f(x_1, x_2, -d),$$

$$\langle f \rangle_d(x_1, x_2) = \frac{1}{2}\{f(x_1, x_2, d) + f(x_1, x_2, -d)\}.$$

Then for a product of two functions $f(x_1, x_2, x_3) g(x_1, x_2, x_3)$ one has

$$[fg]_d(x_1, x_2) = [f]_d(x_1, x_2)\langle g \rangle_d(x_1, x_2) + \langle f \rangle_d(x_1, x_2)[g]_d(x_1, x_2),$$

and, therefore, (3.7) can be written in the form

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \sum_{i=1}^{3} \{[u_i'']_d(x_1 - x_1', x_2 - x_2')\langle \sigma_{i3}(\mathbf{u})\rangle_d(x_1, x_2) + \langle u_i''\rangle_d(x_1 - x_1', x_2 - x_2')[\sigma_{i3}(\mathbf{u})]_d(x_1, x_2) - [\sigma_{i3}'']_d(x_1 - x_1', x_2 - x_2')\langle u_i \rangle_d(x_1, x_2) - \langle \sigma_{i3}'' \rangle_d(x_1 - x_1', x_2 - x_2')[u_i]_d(x_1, x_2)\} = 0. \quad (3.8)$$

It is assumed here that $\varepsilon \ll d$, and then the identity (3.8) holds for the stresses and displacements $\boldsymbol{\sigma}$, $\mathbf{u}$ which solve the given crack problem. Furthermore, (3.8) can be expanded in a power series in ε as $\varepsilon \to 0$, keeping d fixed. This yields, to first order in ε,

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \sum_{i=1}^{3} \{[u_i'']_d(x_1 - x_1', x_2 - x_2')\langle \sigma_{i3}^{(1)}\rangle_d(x_1, x_2) + \langle u_i''\rangle_d(x_1 - x_1', x_2 - x_2')[\sigma_{i3}^{(1)}]_d(x_1, x_2) - [\sigma_{i3}'']_d(x_1 - x_1', x_2 - x_2')\langle u_i^{(1)} \rangle_d(x_1, x_2) - \langle \sigma_{i3}'' \rangle_d(x_1 - x_1', x_2 - x_2')[u_i^{(1)}]_d(x_1, x_2)\} = 0. \quad (3.9)$$

Here $\mathbf{u}^{(1)}$ is the coefficient of ε in the outer expansion

$$\mathbf{u}(\mathbf{x}, \varepsilon) \sim \mathbf{u}^{(0)}(\mathbf{x}) + \varepsilon \mathbf{u}^{(1)}(\mathbf{x}),$$

of the solution $\mathbf{u}(\mathbf{x}, \varepsilon)$; the notation $\sigma_{ij}^{(1)} := \sigma_{ij}(\mathbf{u}^{(1)})$ is adopted.

Equivalently, in terms of $\mathbf{u}'$, $\boldsymbol{\sigma}'$

$$-[\mathbf{u}']_d^t * \langle \boldsymbol{\sigma}^{(1)} \rangle_d + \langle \mathbf{u}' \rangle_d^t * [\boldsymbol{\sigma}^{(1)}]_d + [\mathbf{u}^{(1)}]_d^t * \langle \boldsymbol{\sigma}' \rangle_d - \langle \mathbf{u}^{(1)} \rangle_d^t * [\boldsymbol{\sigma}']_d = 0. \quad (3.10)$$

Here $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}'$ denote the column vectors with components $\sigma_{i3}^{(1)}$, σ_{i3}'. The superscript t means transpose and $*$ denotes convolution with respect to x_1 and x_2. As $d \to +0$, we use the following notations

$$[f] = \lim_{d \to +0} [f]_d, \quad \langle f \rangle = \lim_{d \to +0} \langle f \rangle_d.$$

Now, by writing three linearly independent solutions side by side, $\mathbf{u}'$, $\boldsymbol{\sigma}'$ can be replaced by matrices $\mathbf{U}$, $\boldsymbol{\Sigma}$. The columns of the matrix $\mathbf{U}$ satisfy equilibrium equations with zero body force. The matrix function $\mathbf{U}$ is selected (see the static limit in Willis and Movchan, 1995 and Movchan and Willis, 1995) to be homogeneous of degree $-3/2$, with a discontinuity across the half-plane

$$\{\mathbf{x} : x_3 = 0, 0 < x_1 < +\infty, -\infty < x_2 < +\infty\},$$

so that

$$[\mathbf{U}](x_1, x_2) \sim \left(\frac{2}{\pi x_1}\right)^{1/2} H(x_1)\delta(x_2)\mathbf{I}, \quad \text{as } x_1 \to +0. \tag{3.11}$$

The components of the traction matrix $\mathbf{\Sigma}$ are continuous, and equal to zero on the half-plane across which $\mathbf{U}$ jumps:

$$\mathbf{\Sigma}(x_1, x_2, 0) = 0 \quad \text{for all } x_1 > 0.$$

Then, as $d \to +0$, the identity (3.10) implies

$$-[\mathbf{U}]' * \langle \boldsymbol{\sigma}^{(1)} \rangle + \langle \mathbf{U} \rangle' * [\boldsymbol{\sigma}^{(1)}] + [\mathbf{u}^{(1)}]' * \langle \mathbf{\Sigma} \rangle = 0. \tag{3.12}$$

In particular, when $x_1' > 0$

$$-[\mathbf{U}]' * \langle \boldsymbol{\sigma}^{(1)} \rangle + \langle \mathbf{U} \rangle' * [\boldsymbol{\sigma}^{(1)}] = 0, \tag{3.13}$$

since $([\mathbf{u}^{(1)}]' * \langle \mathbf{\Sigma} \rangle)(x_1', x_2') = 0$ for positive x_1'.

The identity (3.13) applies, in particular, when $x_1' \to +0$. Then, it requires knowledge of the traction vector $\boldsymbol{\sigma}^{(1)}$ on S_0 and of its asymptotic form on the plane $x_3 = 0$, as $x_1 \to +0$. The explicit calculations to follow will show how this delivers expressions for the perturbation ΔK_j to the stress intensity factors.

The explicit formula for $[\mathbf{U}]$ is recorded in Appendix B. The matrix-function $\langle \mathbf{U} \rangle$ that appears in (3.12) was not discussed by Willis and Movchan but, as shown in Appendix B, it is related in a simple way to $[\mathbf{U}]$. Both $[\mathbf{U}]$ and $\langle \mathbf{U} \rangle$ are closely related to the classical weight functions of Bueckner (1987). Similarities and differences, including misprints in Bueckner's formulae, are discussed in Appendix B.

3.3. *Projection of the stress field on the reference plane*

Now, we perform the calculations necessary to evaluate $\boldsymbol{\sigma}^{(1)}$ on the reference plane $x_3 = 0$ on S_0, and just ahead of the crack.

Formally, outside a neighbourhood of the crack front one can derive the following formulae for the traction components on the half plane $x_1 < 0$

$$(\sigma_{\pm}^{(1)})_{3i} := \sigma_{3i}(\mathbf{u}^{(1)}(x_1, x_2, \pm 0)) = \sum_{j=1}^{2} \frac{\partial}{\partial x_j}(\psi(x_1, x_2)\sigma_{ji}(\mathbf{u}^{(0)}(x_1, x_2, \pm 0))), \tag{3.14}$$

where the stress components $\sigma_{ij}^{(0)}$ evaluated on $\mathbf{u}^{(0)}$ are supposed to be given.[1] It follows, since the faces of the unperturbed crack are the traction free, that

$$(\sigma_{\pm}^{(1)})_{33} = 0.$$

Note that the principal part (with respect to r) of (3.14) has a different sign on the upper and lower crack faces.

The local asymptotic representation of the stress components near the crack front is given by the well-known formulae (see, for example, Sih and Liebowitz, 1968)

[1] In particular, if the unperturbed stress state is two-dimensional, on the crack surface on has $\sigma_{12}^{(0)} = 0$, $\sigma_{22}^{(0)} = \nu(\sigma_{11}^{(0)} + \sigma_{33}^{(0)})$ and $\sigma_{11}^{(0)} = \sigma_{33}^{(0)} + \sigma_{11}^{(nc)}$.

$$\sigma_{11} \sim \frac{1}{4\sqrt{\pi r}}\left\{K_{\rm I}\left(3\cos\frac{\theta}{2}+\cos\frac{5\theta}{2}\right)-K_{\rm II}\left(7\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}\right)\right\},$$

$$\sigma_{22} \sim \frac{2\nu}{\sqrt{2\pi r}}\left\{K_{\rm I}\cos\frac{\theta}{2}-K_{\rm II}\sin\frac{\theta}{2}\right\},$$

$$\sigma_{33} \sim \frac{1}{4\sqrt{2\pi r}}\left\{K_{\rm I}\left(5\cos\frac{\theta}{2}-\cos\frac{5\theta}{2}\right)-K_{\rm II}\left(\sin\frac{\theta}{2}-\sin\frac{5\theta}{2}\right)\right\},$$

$$\sigma_{23} \sim \frac{K_{\rm III}}{\sqrt{2\pi r}}\cos\frac{\theta}{2},$$

$$\sigma_{13} \sim \frac{1}{4\sqrt{2\pi r}}\left\{-K_{\rm I}\left(\sin\frac{\theta}{2}-\sin\frac{5\theta}{2}\right)+K_{\rm II}\left(3\cos\frac{\theta}{2}+\cos\frac{5\theta}{2}\right)\right\},$$

$$\sigma_{12} \sim -\frac{K_{\rm III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}. \quad (3.15)$$

Clearly, for the case of a perturbed crack front one can use the formulae (3.15) in the local coordinate system; however the orientation of the system may change according to the change of the crack front, and the origin may be shifted as well.

As mentioned, in this paper we analyze the out-of-plane deflection of the crack front, with the in-plane perturbation analysis being regarded as known (see, for example, Gao and Rice, 1989; Willis and Movchan, 1995; Movchan and Willis, 1995). Thus, the perturbation includes two main parts:

- Superposition of rotations with respect to the x_2-axis and x_1-axis. To leading order approximation, the new local coordinates (x_1'', x_2'', x_3'') and the original coordinates are related by

$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & \varepsilon\omega \\ 0 & 1 & \varepsilon\gamma \\ -\varepsilon\omega & -\varepsilon\gamma & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

  where $0 < \varepsilon \ll 1$, and $\varepsilon\gamma$, $\varepsilon\omega$ are small angles of rotation about the x_1 and x_2 axes.
- Out-of-plane shift (with the same orientation of the axes): the new local coordinates are specified by

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = x_3 - \varepsilon\psi.$$

  Here ψ denotes the deflection of the crack front along the x_3-axis.

Considering first a local rotation, one can write the following relations for the stress components:

$$\begin{aligned} \sigma_{33} &\simeq \sigma_{33}'' + 2\varepsilon\omega\sigma_{13}'' + 2\varepsilon\gamma\sigma_{23}'', \\ \sigma_{13} &\simeq \sigma_{13}'' + \varepsilon\omega(\sigma_{11}'' - \sigma_{33}'') + \varepsilon\gamma\sigma_{12}'' \\ \sigma_{23} &\simeq \sigma_{23}'' + \varepsilon\gamma(\sigma_{22}'' - \sigma_{33}'') + \varepsilon\omega\sigma_{12}''. \end{aligned} \quad (3.16)$$

The orientation of the axes of the coordinate system (x_1'', x_2'', x_3'') corresponds to the orientation of the actual perturbed crack front, and, therefore one can use the asymptotic formulae (3.15) for components σ_{ij}''. Then, we evaluate the stress on the reference plane $x_3 = 0$ for $x_1 > 0$ (one has to set $\theta = -\varepsilon\omega$):

$$\sigma_{13} \simeq \frac{1}{\sqrt{2\pi x_1}}\left\{-\frac{\varepsilon\omega}{2}K_{\rm I}+K_{\rm II}\right\}+O(\varepsilon^2),$$

$$\sigma_{23} \simeq \frac{K_{\rm III}}{\sqrt{2\pi x_1}}+O(\varepsilon^2),$$

$$\sigma_{33} \simeq \frac{1}{\sqrt{2\pi x_1}}\left\{K_{\rm I}-\frac{\varepsilon\omega}{2}K_{\rm II}\right\}+O(\varepsilon^2),$$

$$\sigma_{11} \simeq \frac{1}{\sqrt{2\pi x_1}}\left\{K_{\rm I}+\frac{3\varepsilon\omega}{2}K_{\rm II}\right\}+O(\varepsilon^2),$$

$$\sigma_{22} \simeq \frac{2\nu}{\sqrt{2\pi x_1}}\left\{K_{\rm I}+\frac{\varepsilon\omega}{2}K_{\rm II}\right\}+O(\varepsilon^2),$$

$$\sigma_{12} \simeq -\frac{\varepsilon\omega}{2}\frac{K_{\rm III}}{\sqrt{2\pi x_1}}+O(\varepsilon^2). \tag{3.17}$$

It follows from (3.16), (3.17) that ahead of the reference crack ($x_1 > 0$, $x_3 = 0$) the traction vector $\sigma = (\sigma_{13}, \sigma_{23}, \sigma_{33})'$ admits the asymptotic representation

$$\sigma \simeq \frac{1}{\sqrt{2\pi x_1}}\{\mathbf{I}+\varepsilon\mathbf{\Omega}(\omega,\gamma)\}\mathbf{K}, \tag{3.18}$$

where $\mathbf{I}$ is the identity matrix, and

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 0 & -\omega/2 \\ 0 & 0 & -\gamma(1-2\nu) \\ 3\omega/2 & 2\gamma & 0 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} K_{\rm II} \\ K_{\rm III} \\ K_{\rm I} \end{pmatrix}.$$

Second, we describe the stress field on the reference plane $x_3 = 0$ ahead of the crack front ($x_1 > 0$) for the case of a perturbation produced by a shift along the x_3-axis.

It is assumed that x_1 is greater than the distance from the crack front to the reference plane (or, equivalently, we are describing the outer field corresponding to the exterior of a neighbourhood of the crack front).

Ahead of the crack, the traction vector has the form

$$\sigma \sim \frac{\mathbf{K}}{\sqrt{2\pi x_1}}+\mathbf{P}+x_1^{1/2}\mathbf{A}, \quad \mathbf{A} = \begin{pmatrix} A_{\rm II} \\ A_{\rm III} \\ A_{\rm I} \end{pmatrix}. \tag{3.19}$$

Upon shift of the crack edge to $\{\mathbf{x} : x_3 = \varepsilon\psi(0, x_2)\}$, the stress field in the vicinity of the crack edge changes, from $\sigma^0_{ij}(x_1, x_2, x_3)$ say, to $\sigma^0_{ij}(x_1, x_2, x_3-\varepsilon\psi)$, plus a further term of order ε which is associated with increments $\Delta\mathbf{K}$, $\Delta\mathbf{P}$ and $\Delta\mathbf{A}$ in the quantities $\mathbf{K}$, $\mathbf{P}$ and $\mathbf{A}$. The traction component σ_{i3} on the plane $x_3 = 0$ can therefore be represented, asymptotically, as $\varepsilon \to 0$,

$$\sigma_{i3} \sim \sigma^0_{i3}(x_1, x_2, -\varepsilon\psi)+\frac{\Delta\mathbf{K}\cdot\mathbf{e}_i}{\sqrt{2\pi x_1}}+\Delta P_i+x_1^{1/2}\Delta\mathbf{A}\cdot\mathbf{e}_i, \tag{3.20}$$

where $\mathbf{e}_i$ is the unit vector whose j-component is equal to δ_{ij}. It follows, upon expanding σ^0_{i3} to first order in ε, that

$$\sigma_{i3}(x_1,x_2,0) \sim \sigma_{i3}^0(x_1,x_2,0) - \varepsilon\psi\frac{\partial\sigma_{i3}^0}{\partial x_3}(x_1,x_2,0) + \frac{\Delta\mathbf{K}\cdot\mathbf{e}_i}{\sqrt{2\pi x_1}} + \Delta P_i + x_1^{1/2}\Delta\mathbf{A}\cdot\mathbf{e}_i$$

$$\sim \varepsilon\psi\left[\frac{\partial\sigma_{i1}^0}{\partial x_1}(x_1,x_2,0) + \frac{\partial\sigma_{i2}^0}{\partial x_2}(x_1,x_2,0)\right] + \frac{\mathbf{K}\cdot\mathbf{e}_i}{\sqrt{2\pi x_1}} + P_i + x_1^{1/2}\mathbf{A}\cdot\mathbf{e}_i. \tag{3.21}$$

Subsequent calculations require only the terms that are singular as $x_1 \to 0$. Thus, changes in P_i and A_i can be neglected, and the only important contribution from $\partial\sigma_{i2}^0/\partial x_2$ is that associated with the "**K**-field" (3.15). However, allowance has to be made both for the "**K**-field" $\sigma_{ij}^{(0,K)}$ and the "**A**-field" $\sigma_{ij}^{(0,A)}$ in considering $\partial\sigma_{i1}^0/\partial x_1$ and $\partial\sigma_{i2}^0/\partial x_2$. It is verified directly that the following asymptotic relations hold

$$\frac{\partial}{\partial x_1}\begin{pmatrix}\sigma_{11}^{(0,K)}\\ \sigma_{21}^{(0,K)}\\ \sigma_{31}^{(0,K)}\end{pmatrix} \sim -\frac{1}{2x_1}\begin{pmatrix}\sigma_{11}^{(0,K)}\\ \sigma_{21}^{(0,K)}\\ \sigma_{31}^{(0,K)}\end{pmatrix} = -\frac{1}{\sqrt{\pi}(2x_1)^{3/2}}\begin{pmatrix}K_{\rm I}\\ 0\\ K_{\rm II}\end{pmatrix},$$

$$\frac{\partial}{\partial x_2}\begin{pmatrix}\sigma_{12}^{(0,K)}\\ \sigma_{22}^{(0,K)}\\ \sigma_{32}^{(0,K)}\end{pmatrix} \sim \frac{1}{\sqrt{2\pi x_1}}\begin{pmatrix}0\\ 2\nu K'_{\rm I}\\ K'_{\rm III}\end{pmatrix},$$

$$\frac{\partial}{\partial x_1}\begin{pmatrix}\sigma_{11}^{(0,A)}\\ \sigma_{21}^{(0,A)}\\ \sigma_{31}^{(0,A)}\end{pmatrix} \sim \frac{1}{2x_1}\begin{pmatrix}\sigma_{11}^{(0,A)}\\ \sigma_{21}^{(0,A)}\\ \sigma_{31}^{(0,A)}\end{pmatrix} = \frac{1}{2x_1}\begin{pmatrix}x_1^{1/2}A_{\rm I}\\ \sigma_{21}^{(0,A)}\\ x_1^{1/2}A_{\rm II}\end{pmatrix}.$$

Note that $\sigma_{21}^{(0,A)}$ vanishes for the 2-D loading; in general, it is not zero.[2] The result is that

$$\boldsymbol{\sigma} \simeq \frac{1}{\sqrt{2\pi x_1}}\left\{\mathbf{I} - \frac{\varepsilon\psi^*}{2x_1}\boldsymbol{\Theta}\right\}\mathbf{K} + \mathbf{P} + x_1^{1/2}\mathbf{A} - \frac{\varepsilon\psi^*}{2x_1^{1/2}}\mathbf{G}, \tag{3.22}$$

where $\psi^* = \psi(0, x_2)$, **I** is the identity matrix, **A** and **P** are known vectors, and

$$\boldsymbol{\Theta} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad \mathbf{G} = -\begin{pmatrix}A_{\rm I}\\ (2/\pi)^{1/2}K'_{\rm I}\\ A_{\rm II} + (2/\pi)^{1/2}K'_{\rm III}\end{pmatrix}. \tag{3.23}$$

In Appendix C we analyze the second order expansion [formula (C.1)] of the displacement field near the edge of the unperturbed crack and evaluate components of the vectors **A** and **G** via the coefficients of this expansion.

3.4. *Evaluation of the stress intensity factors*

The effective traction vector $\boldsymbol{\sigma}^{(1)}$ with components $\sigma_{i3}^{(1)}$ can be split into two parts,

$$\boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}_-^{(1)} + \boldsymbol{\sigma}_+^{(1)}.$$

The term $\boldsymbol{\sigma}_-^{(1)}$ has support in the half-plane $x_1 < 0$ and is given by (3.14), and $\boldsymbol{\sigma}_+^{(1)} = \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}_-^{(1)}$. One can evaluate the convolutions in (3.13) when $x'_1 > 0$. Stress components are continuous ahead of the crack, even for the perturbed problem, and

[2] It follows directly from expansion (C.1) that $\sigma_{12}^{(0,A)} = x_1^{1/2}K'_{\rm I}\{W_r^{(1)}(0) - \frac{3}{2}\Upsilon_2^{(1)}(0)\} = \sqrt{(2x_1/\pi)}(1-2\nu)K'_{\rm I}$.

$$[\mathbf{\Sigma}]_d \to 0 \quad \text{as } d \to 0.$$

Thus, the identity (3.13) implies[3]

$$\int_{-\infty}^{x'_1} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 [\mathbf{U}](x'_1 - x_1, x'_2 - x_2)\langle \boldsymbol{\sigma}^{(1)} \rangle (x_1, x_2) - \int_{-\infty}^{0} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \langle \mathbf{U} \rangle' (x'_1 - x_1, x'_2 - x_2)[\boldsymbol{\sigma}^{(1)}](x_1, x_2) = 0. \quad (3.24)$$

This is true for all $x'_1 > 0$ and can be evaluated, in particular, in the limit $x'_1 \to +0$. The notation $\mathbf{I}_j$, $j = 1, 2$, will be used for the integral terms in the left-hand side, so that (3.24) can be represented as

$$\mathbf{I}_1 - \mathbf{I}_2 = 0, \quad (3.25)$$

where

$$\mathbf{I}_1 = \mathbf{I}_1^{(+)} + \mathbf{I}_1^{(-)},$$

and

$$\mathbf{I}_1^{(+)} = \int_0^{x'_1} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 [\mathbf{U}](x'_1 - x_1, x'_2 - x_2)\langle \boldsymbol{\sigma}_+^{(1)} \rangle (x_1, x_2),$$

$$\mathbf{I}_1^{(-)} = \int_{-\infty}^{0} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 [\mathbf{U}](x'_1 - x_1, x'_2 - x_2)\langle \boldsymbol{\sigma}_-^{(1)} \rangle (x_1, x_2).$$

The limiting value as $x'_1 \to +0$ of the integral $\mathbf{I}_1^{(+)}$ only requires the asymptotic representation of the stress field ahead of the crack. The term $\mathbf{I}_1^{(-)}$ depends on the morphology of the crack surface and on tractions applied on the crack faces.

Formally, it follows from (3.14) that the leading order (with respect to r) part in the outer expansion of the stress $\boldsymbol{\sigma}_-^{(1)}$ near the crack edge corresponds to a non-symmetric load. It is taken into account by evaluating the second integral term in (3.25)

$$\mathbf{I}_2 = \langle \mathbf{U} \rangle' * [\boldsymbol{\sigma}_-^{(1)}]. \quad (3.26)$$

It should be mentioned that in the case when the unperturbed state corresponds to the Mode-I crack, the term $\mathbf{I}_2$ vanishes.

A convenient method for evaluating $\mathbf{I}_1^+$ is to employ Fourier transforms: the convolution becomes a product of the transform images and the asymptotic approximation as $x'_1 \to 0$ follows from the behaviour of the transform as $\xi_1 \to \infty$ (ξ_1, ξ_2 denote the transform variables corresponding to x_1, x_2).[4]

Fourier transformation of the sum of (3.18) and (3.22) gives, for the general out-of-plane perturbation,

$$\varepsilon \overline{\boldsymbol{\sigma}_+^{(1)}} \sim \left(\frac{\mathrm{i}}{2}\right)^{1/2} \frac{1}{(\xi_1 + 0\mathrm{i})^{1/2}} \left(\overline{\Delta \mathbf{K}} - \mathrm{i}\varepsilon\xi_1 \overline{\psi * \mathbf{\Theta K}}\right) + \varepsilon \overline{\mathbf{\Omega K}} + \frac{\mathrm{i}^{1/2}\varepsilon\sqrt{\pi}}{2(\xi_1 + 0\mathrm{i})^{1/2}} \overline{\psi * \mathbf{G}}. \quad (3.27)$$

The convention is adopted here that an overline implies the Fourier transform with respect to whatever arguments the original function contains. Thus, $\bar{\boldsymbol{\sigma}}_+^{(1)}$ is a transform with respect

[3] Equation (B.4) shows that [**U**] is symmetric; hence, transposition of this term is redundant.

[4] Here we use the notations similar to Willis and Movchan (1995). In particular, for a function $f(x)$ its Fourier transform is specified by $\bar{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{\mathrm{i}\xi x}\, \mathrm{d}x$.

to x_1, x_2 and is a function of ξ_1, ξ_2, whereas the terms on the right, such as $\overline{\Delta\mathbf{K}}$, are transforms with respect to x_2 only and are functions of ξ_2. The Fourier transform of the weight function is characterised by the following asymptotic formula (see Willis and Movchan, 1995; Movchan and Willis, 1995)

$$\overline{[\mathbf{U}]} \sim (2\mathrm{i})^{1/2}\frac{1}{(\xi_1+0\mathrm{i})^{1/2}}\left\{\mathbf{I}+\frac{\mathrm{i}}{\xi_1+0\mathrm{i}}\bar{\mathbf{Q}}\right\}, \tag{3.28}$$

where

$$\bar{\mathbf{Q}} = \begin{bmatrix} -\dfrac{2-3\nu}{2(2-\nu)}|\xi_2| & \dfrac{\mathrm{i}\xi_2\nu}{2-\nu} & 0 \\ \dfrac{\mathrm{i}\xi_2\nu}{2-\nu} & -\dfrac{2+\nu}{2(2-\nu)}|\xi_2| & 0 \\ 0 & 0 & -\dfrac{1}{2}|\xi_2| \end{bmatrix}.$$

It follows from (3.27), (3.28) that as $|\xi_1| \to +\infty$, to order $O(1/|\xi_1|)$

$$\varepsilon\overline{[\mathbf{U}]'\boldsymbol{\sigma}_+^{(1)}} \sim \varepsilon\overline{\psi^*\boldsymbol{\Theta}\mathbf{K}}+\frac{\mathrm{i}}{\xi_1+0\mathrm{i}}\left\{\overline{\Delta\mathbf{K}}+\varepsilon\overline{\boldsymbol{\Omega}\mathbf{K}}+\varepsilon\bar{\mathbf{Q}}\overline{\psi^*\boldsymbol{\Theta}\mathbf{K}}+\varepsilon\left(\frac{\pi}{2}\right)^{1/2}\overline{\psi^*\mathbf{G}}\right\}. \tag{3.29}$$

Hence, by inverting (3.29),

$$\varepsilon\mathbf{I}_1^{(+)} = \varepsilon[\mathbf{U}]' * \boldsymbol{\sigma}_+^{(1)} \sim \varepsilon\psi^*\boldsymbol{\Theta}\mathbf{K}\delta(x_1) + \left\{\Delta\mathbf{K}+\varepsilon\mathbf{Q}*(\psi^*\boldsymbol{\Theta}\mathbf{K})+\varepsilon\left(\frac{\pi}{2}\right)^{1/2}\psi^*\mathbf{G}+\varepsilon\boldsymbol{\Omega}\mathbf{K}\right\}H(x_1), \tag{3.30}$$

where the convolution is considered with respect to x_1 and x_2. Here,

$$\mathbf{Q} = \frac{1}{\pi}\begin{bmatrix} \dfrac{2-3\nu}{2(2-\nu)} & 0 & 0 \\ 0 & \dfrac{2+\nu}{2(2-\nu)} & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\frac{1}{x_2^2}+\frac{\nu}{2-\nu}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\delta'(x_2), \tag{3.31}$$

and the matrix $\boldsymbol{\Omega}$ has the same form as in (3.18) with $\omega = \psi_{,1}$, and $\gamma = \psi_{,2}$ (the representation for the matrix $\boldsymbol{\Omega}$ agrees with the analysis given by Gao, 1992). Since $x_1' \to +0$, the delta-function in (3.30) makes no contribution. Hence,

$$\lim_{x_1'\to+0}\varepsilon\mathbf{I}_1^{(+)} = \Delta\mathbf{K}+\varepsilon\mathbf{Q}*(\psi^*\boldsymbol{\Theta}\mathbf{K})+\varepsilon\left(\frac{\pi}{2}\right)^{1/2}\psi^*\mathbf{G}+\varepsilon\boldsymbol{\Omega}\mathbf{K}.$$

Next, we express the "morphology" term $\mathbf{I}_1^{(-)}$ in the form

$$
-\lim_{x'_1\to+0}\varepsilon\mathbf{I}_1^{(-)} = \begin{pmatrix} \Delta K_{\mathrm{II}}^{(\mathrm{morphology})}(x'_2) \\ \Delta K_{\mathrm{III}}^{(\mathrm{morphology})}(x'_2) \\ 0 \end{pmatrix} \tag{3.32}
$$

with

$$
\Delta K_j^{(\mathrm{morphology})}(x'_2) = -\varepsilon\int_{-\infty}^{0} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \left\{ \sum_{k=1}^{2} h_k^{(j)}(x_1, x_2 - x'_2) \times \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(\psi(x_1,x_2)\langle\sigma_{ik}(\mathbf{u}^{(0)})\rangle(x_1,x_2))\right\}, \quad j = \mathrm{II}, \mathrm{III}. \tag{3.33}
$$

Here

$$
h_k^{(\mathrm{II})}(x_1,x_2) = [U_{k1}](-x_1,-x_2), h_k^{(\mathrm{III})}(x_1,x_2) = [U_{k2}](-x_1,-x_2).
$$

The explicit form of [U] is given by (B.5). It may be noted, from (3.15), that the averages $\langle\cdot\rangle$ of the singular part of the stress components $\sigma_{ij}(\mathbf{u}^0)$ are zero.

Finally, we analyze the integral $\mathbf{I}_2$ from (3.25). As $x'_1 \to +0$ it admits the representation

$$
\mathbf{I}_2 \sim -\langle\mathbf{U}\rangle' * [\boldsymbol{\sigma}_-^{(1)}]. \tag{3.34}
$$

The convolution has a singularity consistent with an "inner limit" ($d\to 0$) of taking an "outer expansion", and it is to be interpreted in the sense of generalized functions. The details of the calculations are given in Appendix B. It follows immediately from the form (B.7) for $\overline{\langle\mathbf{U}\rangle}$ and the relation $(\sigma_-^{(1)})_{33} = 0$, that the first two components of the vector term $\langle\mathbf{U}\rangle' * [\boldsymbol{\sigma}_-^{(1)}]$ are identically zero. We adopt the following notation

$$
\begin{pmatrix} 0 \\ 0 \\ \Delta K_{\mathrm{I}}^{(\mathrm{skew})} \end{pmatrix} := \varepsilon \lim_{x_1\to+0} \langle\mathbf{U}\rangle' * [\boldsymbol{\sigma}_-^{(1)}]. \tag{3.35}
$$

Hence, (3.25), (3.30), (3.32) and (3.34) imply

$$
\Delta\mathbf{K} \sim -\varepsilon\left\{\mathbf{Q} * (\psi * \boldsymbol{\Theta}\mathbf{K}^{(0)}) - \left(\frac{\pi}{2}\right)^{1/2}\psi * \mathbf{G} + \boldsymbol{\Omega}(\omega,\gamma)\mathbf{K}^{(0)}\right\} + \begin{pmatrix} \Delta K_{\mathrm{II}}^{(\mathrm{morphology})} \\ \Delta K_{\mathrm{III}}^{(\mathrm{morphology})} \\ \Delta K_{\mathrm{I}}^{(\mathrm{skew})} \end{pmatrix}. \tag{3.36}
$$

3.5. *Comparison with the two-dimensional case*

The results given above for a three-dimensional perturbation of the crack front are consistent with those presented in Section 2. Namely, if in the formula [obtained by Fourier transforming (3.36)]

$$
\overline{\Delta\mathbf{K}} \sim -\varepsilon\{\bar{\mathbf{Q}}(\overline{\psi * \boldsymbol{\Theta}\mathbf{K}^{(0)}}) + \overline{\boldsymbol{\Omega}\mathbf{K}^{(0)}}\} + \begin{pmatrix} \overline{\Delta K_{\mathrm{II}}^{(\mathrm{morphology})}} \\ \overline{\Delta K_{\mathrm{III}}^{(\mathrm{morphology})}} \\ \overline{\Delta K_{\mathrm{I}}^{(\mathrm{skew})}} \end{pmatrix}
$$

we let $\xi_2 = 0$ and $\gamma = \varepsilon\,\partial\psi/\partial x_2 = 0$, inversion of the terms $\bar{\mathbf{Q}}$ and $\overline{\langle\mathbf{U}\rangle}$ reduce to zero and the remaining Fourier transform with respect to ξ_1 yields

$$\begin{pmatrix}\Delta K_{II}\\ \Delta K_{I}\end{pmatrix} = -\varepsilon\left\{\begin{pmatrix}0 & -\omega/2\\ 3\omega/2 & 0\end{pmatrix}\begin{pmatrix}K_{II}^{(0)}\\ K_{I}^{(0)}\end{pmatrix} - \left(\frac{\pi}{2}\right)^{1/2}\psi^*\mathbf{G} + \begin{pmatrix}\Delta K_{II}^{(\mathrm{morphology})}\\ 0\end{pmatrix}\right\}. \tag{3.37}$$

This agrees with the formulae (2.32), (2.46), with

$$\Delta K_{II}^{(\mathrm{morphology})} = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}\frac{1}{\sqrt{-x_1}}\frac{\partial}{\partial x_1}\{\psi(x_1)\sigma_{11}(\mathbf{u}^{(0)};x_1,0)\}\,\mathrm{d}x_1,$$

and

$$\mathbf{G} = -\begin{pmatrix}A_{I}\\ 0\\ A_{II}\end{pmatrix}.$$

3.6. *Example: sinusoidal perturbation of the crack surface*

Here we consider the simple example of a sinusoidal perturbation of the crack surface. This example was treated by several authors (see Gao, 1992; Xu *et al.*, 1994; Ball and Larralde, 1995), and, apparently, the results of calculations presented in these papers show some disagreement.

Namely, we assume that[5]

$$\psi = \psi(x_2) = \mathscr{A}\cos(kx_2), \tag{3.38}$$

and, consequently

$$\omega = 0, \quad \gamma = \psi'(x_2) = -\mathscr{A}k\sin(kx_2). \tag{3.39}$$

Also, it is assumed that the stress-intensity factors, corresponding to the unperturbed state, are x_2-independent. In this case

$$\mathbf{Q}*(\psi^*\boldsymbol{\Theta}\mathbf{K}) = -\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{\psi'(x_2')}{x_2-x_2'}\begin{bmatrix}\frac{2-3\nu}{2(2-\nu)}K_I^{(0)}\\ 0\\ \frac{1}{2}K_{II}^{(0)}\end{bmatrix}\mathrm{d}x_2' - \frac{\nu}{2-\nu}\psi'(x_2)\begin{pmatrix}0\\ K_I^{(0)}\\ 0\end{pmatrix}$$

$$= -\mathscr{A}k\cos(kx_2)\begin{bmatrix}\frac{2-3\nu}{2(2-\nu)}K_I^{(0)}\\ 0\\ \frac{1}{2}K_{II}^{(0)}\end{bmatrix} + \frac{\nu}{2-\nu}\mathscr{A}k\sin(kx_2)\begin{pmatrix}0\\ K_I^{(0)}\\ 0\end{pmatrix},$$

and the formula (3.36) is reduced to

[5] In formula (3.38) the sign of k is not important, but further in the text we assume that k is positive.

$$\Delta\mathbf{K} = \begin{pmatrix} \Delta K_{\mathrm{II}} \\ \Delta K_{\mathrm{III}} \\ \Delta K_{\mathrm{I}} \end{pmatrix} \sim -\varepsilon\left\{ -\mathscr{A}k\cos(kx_2)\begin{bmatrix} \dfrac{2-3\nu}{2(2-\nu)}K_{\mathrm{I}}^{(0)} \\ 0 \\ \dfrac{1}{2}K_{\mathrm{II}}^{(0)} \end{bmatrix} + \frac{2(1-\nu)^2}{2-\nu}k\mathscr{A}\sin(kx_2)\begin{pmatrix} 0 \\ K_{\mathrm{I}}^{(0)} \\ 0 \end{pmatrix} \right.$$

$$\left. -2k\mathscr{A}\sin(kx_2)\begin{pmatrix} 0 \\ 0 \\ K_{\mathrm{III}}^{(0)} \end{pmatrix} + \left(\frac{\pi}{2}\right)^{1/2}\psi\mathbf{G}\right\} + \begin{pmatrix} \Delta K_{II}^{(\mathrm{morphology})} \\ \Delta K_{\mathrm{III}}^{(\mathrm{morphology})} \\ \Delta K_{\mathrm{I}}^{(\mathrm{skew})} \end{pmatrix}. \quad (3.40)$$

When the perturbation function ψ is independent of x_1 (as in the present example) the representation (3.35) for $\Delta K_{\mathrm{I}}^{(\mathrm{skew})}$ takes the form (see Appendix B)

$$\Delta K_{\mathrm{I}}^{(\mathrm{skew})}(x_2) = \varepsilon\frac{1-2\nu}{\sqrt{2}(1-\nu)}\left\{\frac{\partial}{\partial x_2}(\psi(x_2)K_{\mathrm{III}}^{(0)}) + \frac{1-\nu}{\pi}\int_{-\infty}^{\infty}\frac{\partial(\psi(x_2')K_{\mathrm{II}}^{(0)})/\partial x_2'}{x_2'-x_2}\,\mathrm{d}x_2'\right\}. \quad (3.41)$$

In particular, for constant $K_{\mathrm{II}}^{(0)}$, $K_{\mathrm{III}}^{(0)}$ it reduces to

$$\Delta K_{\mathrm{I}}^{(\mathrm{skew})}(x_2) = -\varepsilon\frac{1-2\nu}{\sqrt{2}(1-\nu)}\mathscr{A}\{K_{\mathrm{III}}^{(0)}k\sin(kx_2) + (1-\nu)K_{\mathrm{II}}^{(0)}k\cos(kx_2)\}. \quad (3.42)$$

In addition, let us represent the components $\langle\sigma_{ij}(\mathbf{u}^{(0)})\rangle$, $i,j = 1,2$, in the form

$$\langle\sigma_{ij}(\mathbf{u}^{(0)})\rangle = t_{ij}^{(1)} + T_{ij},$$

where T_{ij}, $i,j = 1,2$, are constant T-stresses applied at infinity.

Then the quantities $K_j^{(\mathrm{morphology})}$, $j = \mathrm{II},\mathrm{III}$, are evaluated as

$$\Delta K_j^{(\mathrm{morphology})}(x_2') = \varepsilon\int_{-\infty}^{0}\mathrm{d}x_1\int_{-\infty}^{\infty}\mathrm{d}x_2\left\{\sum_{k=1}^{2}h_k^{(j)}(x_1,x_2-x_2')\right.$$

$$\left.\times\sum_{i=1}^{2}\frac{\partial}{\partial x_1}(\psi(x_1,x_2)t_{ik}^{(1)}(\mathbf{u}^{(0)}(x_1,x_2,+0)))\right\} + \Delta K_j^{(T\text{-stress})},\quad j = \mathrm{II},\mathrm{III}. \quad (3.43)$$

The T-stress terms are specified by

$$\Delta K_{\mathrm{II}}^{(T\text{-stress})}(x_2) = \varepsilon\frac{\mathscr{A}\sqrt{2k}}{2-\nu}\{2T_{21}\sin(kx_2) + \nu T_{22}\cos(kx_2)\}, \quad (3.44)$$

$$\Delta K_{\mathrm{III}}^{(T\text{-stress})}(x_2) = \varepsilon\frac{\mathscr{A}\sqrt{2k}}{2-\nu}\{\nu T_{21}\cos(kx_2) + 2(1-\nu)T_{22}\sin(kx_2)\}. \quad (3.45)$$

When the unperturbed state corresponds to the 2-D loading (independent of x_2), the asymptotic formulae (3.15) are valid everywhere on the crack faces, and the quantities $t_{ik}^{(1)}$ vanish. We remark that Xu *et al.* (1994) have additional morphology terms (with a factor $\sqrt{2}$) in the formulae associated with K_{II} and K_{III}; our calculations show that with the correct choice of the weight functions these terms cancel; also, we note the sign discrepancy in the second term in (3.44).[6]

[6] Xu *et al.* (1994) apparently employed in their calculations the skew symmetry $-h_1^{\mathrm{II}} = h_2^{\mathrm{III}}$ (their notations), though their formula (20) gives $h_1^{\mathrm{II}} = h_2^{\mathrm{III}}$; in Appendix B we show that the weight matrix-function [U] should be symmetric.

We emphasize the presence of the term $(\pi/2)^{1/2}\psi\mathbf{G}$ in (3.36) which is related to the high order expansion for the stress components. This term was not indicated in the papers of Gao (1992), Xu *et al.* (1994), Ball and Larralde (1995).

In comparison with the present work the papers by Xu *et al.* (1994) and by Gao (1992) use the "travelling reference plane" and the local system of coordinates which moves along the crack front. Gao (1992) uses a shift of the reference plane without changing its orientation. In addition, Xu *et al.* (1994) rotate the reference crack in accordance with the orientation of the local system of coordinates on the crack front. Consequently, the coefficients K_j^{∞} (in terms of notations of these papers) are not ε-independent.

4. CONCLUSION

Our intention was to present in an explicit way the asymptotic analysis of the stress intensity factors for a crack whose surface is characterized by a small out-of-plane deflection. Clearly, the out-of-plane shift of the crack front produces a singular perturbation of the boundary value problem, and highly singular terms occur in the formal analysis. We would like to emphasize the importance of the two term asymptotic approximation for the stress field and the weight functions near the crack front.

Even for the 2-D case we have derived formulae which generalize results existing in the literature: first, our formulae cover the case when the T-stress is not constant, and, second, the analysis does not impose any requirement to relocate a local system of coordinates to the crack tip.

The 3-D case is substantially more difficult but we have shown how it can be treated with relative ease using the method introduced by Willis and Movchan (1995). This method uses the concept of generalized functions applied to the Betti identity written in terms of Fourier transforms. As a matter of exercise the reader can try an alternative approach (based on ideas similar to those displayed in Section 2), and one can observe that the two term asymptotic approximations of the stress field and of the weight function are required in a neighbourhood of the edge of the 3-D crack. Then, one can also discover that the analysis of the 3-D edge singularity requires much more work than one related to a conical singularity (the case of two dimensions) discussed in Section 2. Thus, the reader can appreciate the effectiveness of the approach presented in Section 3 (we can derive the asymptotic formulae for the stress-intensity factors just on one page). In addition, we would like to mention that the asymptotic algorithm of Section 3 can be extended to the case of dynamic cracks in three dimensions; this is the subject of a separate paper (Willis and Movchan, 1997).

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APPENDIX A: APPLICATION OF THE BETTI FORMULA IN EVALUATION OF THE STRESS-INTENSITY FACTORS FOR A 2-D SEMI-INFINITE CRACK

Here we clarify the derivation of formulae (2.31) and (2.44) presented in Section 2.

First, consider the case when the unperturbed problem concerns the Mode-I crack. Applying the Betti formula to the vector function $\mathbf{v}^{(1)}$ and the weight function $\boldsymbol{\zeta}^{(\mathrm{II})}$ in the region $B_R\backslash S_0$ [see Section 2, formula (2.24)] we obtain

$$\sum_{k=1}^{3} I_k(R) = 0, \tag{A.1}$$

where

$$I_1(R) = \int_{\{x:\|\mathbf{x}\|=R\}} \{\zeta_r^{(\mathrm{II})}\sigma_{rr}(\mathbf{v}^{(1)};\mathbf{x})+\zeta_\theta^{(\mathrm{II})}\sigma_{r\theta}(\mathbf{v}^{(1)};\mathbf{x})-v_r^{(1)}\sigma_{rr}(\boldsymbol{\zeta}^{(\mathrm{II})};\mathbf{x})-v_\theta^{(1)}\sigma_{r\theta}(\boldsymbol{\zeta}^{(\mathrm{II})};\mathbf{x})\}\,\mathrm{d}l_{\mathbf{x}},$$

$$I_2(R) = -\int_{\{x:\|\mathbf{x}\|=1/R\}} \{\zeta_r^{(\mathrm{II})}\sigma_{rr}(\mathbf{v}^{(1)};\mathbf{x})+\zeta_\theta^{(\mathrm{II})}\sigma_{r\theta}(\mathbf{v}^{(1)};\mathbf{x})-v_r^{(1)}\sigma_{rr}(\boldsymbol{\zeta}^{(\mathrm{II})};\mathbf{x})-v_\theta^{(1)}\sigma_{r\theta}(\boldsymbol{\zeta}^{(\mathrm{II})};\mathbf{x})\}\,\mathrm{d}l_{\mathbf{x}},$$

and

$$I_3(R) = \sum_{\pm} \mp \int_{S_0\cap B_R} \zeta_1^{(\mathrm{II})}\sigma_{12}(\mathbf{v}^{(1)};x_1,\pm 0)\,\mathrm{d}l_{\mathbf{x}}.$$

As $R\to\infty$,

$$I_1(R)\to 0,$$

$$I_2(R)\to -\frac{1}{2}\psi Q_1(0)+\frac{1}{2}\omega K_1(0)-K'_{\mathrm{II}}(0),$$

$$I_3(R)\to D := -2\int_{-\infty}^{0} \zeta_1^{(\mathrm{II})}(x_1,+0)\frac{\partial}{\partial x_1}\{f(x_1)\sigma_{11}(\mathbf{u}^{(0)};x_1,+0)\}\,\mathrm{d}x_1.$$

As a result, we have the formula (2.31).

In a similar way one can apply the Betti formula to the vector functions $\boldsymbol{\zeta}^{(\mathrm{I})}$ and $\mathbf{v}^{(1)}$ and show that

$$K'_{\mathrm{I}}(0) = 0.$$

Now, let us clarify the derivation of the formula (2.44). The formal procedure is similar to what was presented above and, as $R\to\infty$, the Betti formula applied to $\mathbf{v}^{(1)}$ [see (2.39)] and $\boldsymbol{\zeta}^{(\mathrm{II})}$ yields (A.1), where

$$I_1(R)\to -(\psi Q_{\mathrm{II}}(0)+\omega K_{\mathrm{II}}(0)),$$

$$I_2(R)\to \tfrac{1}{2}\psi Q_{\mathrm{II}}(0)-\tfrac{1}{2}\omega K_{\mathrm{II}}(0)-K'_{\mathrm{I}}(0),$$

and

$$I_3(R)\to \mathscr{F}(X) := -2\int_{-\infty}^{0} \zeta_2^{(\mathrm{I})}(x_1,+0)\frac{\partial}{\partial x_1}\{f(x_1)\sigma_{12}(\mathbf{u}^{(0)};x_1,+0)\}\,\mathrm{d}x_1. \tag{A.2}$$

Let us mention that in the formulation presented in Section 2 the supports of the functions $f(x_1)$ and $\sigma_{12}(\mathbf{u}^{(0)};x_1,+0)$ do not intersect and, therefore, the quantity $\mathscr{F}(X)$ vanishes, and as a result we obtain formula (2.44). However, one can extend the applied shear load up to the crack tip including the perturbation region. In this case the term (A.2) will contribute to the final answer; but in addition we would need to deal with the expansion of the load function with respect to ε. Our intention is to make the main text as simple as possible. Therefore, we concentrate on the case where the applied load is independent of ε, and, consequently,

$$\operatorname{mes}\{\operatorname{supp}(f(x_1))\bigcap \operatorname{supp}(\sigma_{12}(\mathbf{u}^{(0)};x_1,+0))\} = 0.$$

Finally, one can apply the Betti formula to $\mathbf{v}^{(1)}$, given by (2.39), and to $\boldsymbol{\zeta}^{(\mathrm{I})}$ in the region $B_R\backslash S_0$. As $R\to\infty$, we deduce (A.1) with

$$I_j \to 0, \quad j = 1, 3; \quad I_2 \to -K'_{II}(0).$$

Consequently, the relation (2.45) holds.

APPENDIX B: SYMMETRIC PART OF THE WEIGHT FUNCTION

The Fourier transform $\overline{\langle U\rangle}$

Apply the identity

$$-[\mathbf{U}]'_d * \langle \boldsymbol{\sigma}\rangle_d + \langle \boldsymbol{\Sigma}\rangle'_d * [\mathbf{u}]_d + \langle \mathbf{U}\rangle'_d * [\boldsymbol{\sigma}]_d - [\boldsymbol{\Sigma}]'_d * \langle \mathbf{u}\rangle_d = 0 \tag{B.1}$$

to the field $\mathbf{u}$ generated by a layer of body force $\mathbf{f}(x_1, x_2)$ distributed on the plane $x_3 = 0$. In this case the displacement $\mathbf{u}$ and the stress $\boldsymbol{\sigma}$ are given by

$$\mathbf{u} = \mathscr{G} * \mathbf{f} \quad \text{and} \quad \boldsymbol{\sigma} = \mathbf{S} * \mathbf{f},$$

where $\mathscr{G}$ is the infinite-body Green's matrix, and $\mathbf{S}$ is the corresponding matrix whose columns represent the traction vectors. In the limit, when $d \to 0$, one has

$$[\mathbf{u}] = 0, \quad [\boldsymbol{\sigma}] = -\mathbf{f} \quad \text{and} \quad [\boldsymbol{\Sigma}] = 0, \tag{B.2}$$

and relation (B.1) reduces to

$$-\langle \mathbf{U}\rangle' * \mathbf{f} = [\mathbf{U}]' * \langle \boldsymbol{\sigma}\rangle = [\mathbf{U}]' * \langle \mathbf{S}\rangle * \mathbf{f}. \tag{B.2}$$

Therefore,

$$\langle \mathbf{U}\rangle' = -[\mathbf{U}]' * \langle \mathbf{S}\rangle$$

or

$$\langle \mathbf{U}\rangle = -\langle \mathbf{S}\rangle' * [\mathbf{U}].$$

In terms of Fourier transforms one has

$$\overline{\langle \mathbf{U}\rangle} = -\overline{\langle \mathbf{S}\rangle}'\overline{[\mathbf{U}]}. \tag{B.3}$$

It follows from the static limit of the results by Willis and Movchan (1995) and Movchan and Willis (1995) that[7]

$$\overline{[\mathbf{U}]} = \frac{(2\mathrm{i})^{1/2}}{(2-\nu)(\xi_1 + \mathrm{i}|\xi_2|)^{1/2}} \begin{bmatrix} 2 - \dfrac{\nu\xi_1}{\xi_1 + \mathrm{i}|\xi_2|} & -\dfrac{\nu\xi_2}{\xi_1 + \mathrm{i}|\xi_2|} & 0 \\ -\dfrac{\nu\xi_2}{\xi_1 + \mathrm{i}|\xi_2|} & 2(1-\nu) + \dfrac{\nu\xi_1}{\xi_1 + \mathrm{i}|\xi_2|} & 0 \\ 0 & 0 & 2-\nu \end{bmatrix}. \tag{B.4}$$

In terms of the original variables x_1, x_2 the matrix function $[\mathbf{U}]$ has the form

$$[\mathbf{U}](x_1, x_2) = \frac{\sqrt{2x_1}H(x_1)}{\pi^{3/2}(x_1^2 + x_2^2)} \begin{bmatrix} 1 + \dfrac{2\nu}{2-\nu}\cos[2\mathscr{P}] & \dfrac{2\nu}{2-\nu}\sin[2\mathscr{P}] & 0 \\ \dfrac{2\nu}{2-\nu}\sin[2\mathscr{P}] & 1 - \dfrac{2\nu}{2-\nu}\cos[2\mathscr{P}] & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.5}$$

where $\mathscr{P} = \tan^{-1}(x_2/x_1)$. Bueckner (1987, pp. 86 and 87) gives formulae which provide the stress intensity factors generated by loading just on the upper face of a semi-infinite plane crack. The basic identity presented in Section 3.2 provides, for the perturbed crack, the representation

$$\mathbf{K}^{(0)}(x'_2) = -[\mathbf{U}] * \langle \boldsymbol{\sigma}^{(0)}\rangle + \langle \mathbf{U}\rangle * [\boldsymbol{\sigma}^{(0)}],$$

evaluated at $(0, x'_2)$. When applied to loading $\mathbf{T}$ on the upper face only, $\langle \boldsymbol{\sigma}^{(0)}\rangle = -\frac{1}{2}\mathbf{T}$ and $[\boldsymbol{\sigma}^{(0)}] = -\mathbf{T}$. Comparison with Bueckner (1987) requires $[\mathbf{U}](x_1, x_2)$ to be replaced by $2^{1/2}\pi^{3/2}[\mathbf{U}](-x_1, -x_2)$ and $\langle \mathbf{U}\rangle(x_1, x_2)$ to be replaced by $-2^{3/2}\pi^{3/2}\langle \mathbf{U}\rangle(-x_1, -x_2)$; in addition, Bueckner's angle λ is equivalent to our $-\mathscr{P}$. These correspondences

[7] The formula (A.5) in the Appendix of the second paper cited has a misprint, an extra factor 2, and it was taken into account when we derived (B.4).

demonstrate agreement between (B.5) and Bueckner (1987), except for the sign of the second column of the matrix.[8]

It is verified by direct calculation that the Fourier transform of the traction matrix is given by

$$\overline{\langle \mathbf{S} \rangle} = -\frac{\mathrm{i}(1-2\nu)}{4(1-\nu)(\xi_1^2+\xi_2^2)^{1/2}}\begin{pmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ -\xi_1 & -\xi_2 & 0 \end{pmatrix}. \tag{B.6}$$

Then, it follows from (B.3) that

$$\overline{\langle \mathbf{U} \rangle} = \mathrm{i}\left(\frac{\mathrm{i}}{2}\right)^{1/2}\frac{1-2\nu}{(2-\nu)(1-\nu)}\frac{(\xi_1-\mathrm{i}|\xi_2|)^{1/2}}{\xi_1^2+\xi_2^2}\begin{pmatrix} 0 & 0 & (2-\nu)\xi_1 \\ 0 & 0 & (2-\nu)\xi_2 \\ (\nu-2)\xi_1-\mathrm{i}\nu|\xi_2| & -2(1-\nu)\xi_2 & 0 \end{pmatrix}. \tag{B.7}$$

The above matrix function can be represented as

$$\overline{\langle \mathbf{U} \rangle} = \overline{\langle \mathbf{U} \rangle_+} + \overline{\langle \mathbf{U} \rangle_-},$$

where $\overline{\langle \mathbf{U} \rangle_+}$ represents the Fourier transform of the "+" function (which vanishes when $x_1 < 0$), and $\overline{\langle \mathbf{U} \rangle_-}$ corresponds to the Fourier transform of the "−" function (which is equal to zero for all $x_1 > 0$). Then, the second term $\mathbf{I}_2$ in (3.25) can be rewritten as

$$\langle \mathbf{U} \rangle'_+ * [\boldsymbol{\sigma}^{(1)}_-] + \langle \mathbf{U} \rangle'_- * [\boldsymbol{\sigma}^{(1)}_-] = \int_{-\infty}^{0} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \langle \mathbf{U} \rangle'_+ (x'_1 - x_1, x'_2 - x_2)[\boldsymbol{\sigma}^{(1)}_-](x_1, x_2), \tag{B.8}$$

and, therefore, the "−" function $\langle \mathbf{U} \rangle'_-$ does not contribute to the final answer for the stress-intensity factors.

One could also predict that components of $\langle \mathbf{U} \rangle_+$ must be equal to components of the classical symmetric weight functions derived by Bueckner (1987).[9]

The matrix $\langle \boldsymbol{U} \rangle_+$

In order to derive the representation for components of $\langle \mathbf{U} \rangle$ we shall analyze the inverse Fourier transforms (with respect to ξ_1 and ξ_2) of the right-hand side of (B.7). This formal analysis involves the following two functions

$$f_j(\xi_1, \xi_2) = \frac{\xi_j}{(\xi_1 - \mathrm{i}|\xi_2|)^{1/2}(\xi_1 + \mathrm{i}|\xi_2|)}, \quad j = 1, 2,$$

which can be split as

$$f_j(\xi_1, \xi_2) = f_j^{(-)}(\xi_1, \xi_2) + f_j^{(+)}(\xi_1, \xi_2), \quad j = 1, 2. \tag{B.9}$$

Here, $f_j^{(+)}$ are the "+" functions analytic in the upper half-plane (in ξ_1), and $f_j^{(-)}$ are the "−" functions which are analytic in the lower half-plane:

$$f_1^{(+)}(\xi_1, \xi_2) = -\frac{\mathrm{i}|\xi_2|}{(\xi_1 + \mathrm{i}|\xi_2|)(-2\mathrm{i}|\xi_2|)^{1/2}},$$

$$f_2^{(+)}(\xi_1, \xi_2) = \frac{\xi_2}{(\xi_1 + \mathrm{i}|\xi_2|)(-2\mathrm{i}|\xi_2|)^{1/2}}, \tag{B.10}$$

and

$$f_1^{(-)}(\xi_1, \xi_2) = \frac{1}{\xi_1 + \mathrm{i}|\xi_2|}\left\{\frac{\xi_1}{(\xi_1 - \mathrm{i}|\xi_2|)^{1/2}} + \frac{\mathrm{i}|\xi_2|}{(-2\mathrm{i}|\xi_2|)^{1/2}}\right\},$$

$$f_2^{(-)}(\xi_1, \xi_2) = \frac{1}{\xi_1 + \mathrm{i}|\xi_2|}\left\{\frac{\xi_2}{(\xi_1 - \mathrm{i}|\xi_2|)^{1/2}} - \frac{\xi_2}{(-2\mathrm{i}|\xi_2|)^{1/2}}\right\}. \tag{B.11}$$

Hence,

$$\overline{\langle \mathbf{U} \rangle_+} = \frac{C}{(\xi_1 + \mathrm{i}|\xi_2|)(-2\mathrm{i}|\xi_2|)^{1/2}}\begin{pmatrix} 0 & 0 & -(2-\nu)\mathrm{i}|\xi_2| \\ 0 & 0 & (2-\nu)\xi_2 \\ 2(1-\nu)\mathrm{i}|\xi_2| & -2(1-\nu)\xi_2 & 0 \end{pmatrix}, \tag{B.12}$$

where

[8] We believe that formula (B.5) is correct.

[9] The paper cited contains misprints in the main formulae for the weight function; this appendix includes an alternative method of derivation of the weight functions and corrected formulae.

$$C = \mathrm{i}\left(\frac{\mathrm{i}}{2}\right)^{1/2}\frac{1-2\nu}{2(2-\nu)(1-\nu)}.$$

Let $\langle\bar{\mathbf{U}}\rangle_+(x_1', \xi_2)$ denote the Fourier transform of $\langle\mathbf{U}\rangle_+$ with respect to x_2' only. Direct calculations show that

$$\langle\bar{\mathbf{U}}\rangle_+(x_1', \xi_2) = -\frac{\mathrm{i}C}{(-2\mathrm{i}|\xi_2|)^{1/2}}e^{-|\xi_2|x_1'}H(x_1')\begin{pmatrix} 0 & 0 & -(2-\nu)\mathrm{i}|\xi_2| \\ 0 & 0 & (2-\nu)\xi_2 \\ 2(1-\nu)\mathrm{i}|\xi_2| & -2(1-\nu)\xi_2 & 0 \end{pmatrix}. \tag{B.13}$$

Then, we introduce two auxiliary functions

$$p_1(x_1', \xi_2) = |\xi_2|^{1/2}e^{-|\xi_2|x_1'},$$

and

$$p_2(x_1', \xi_2) = \frac{\xi_2 e^{-|\xi_2|x_1'}}{|\xi_2|^{1/2}} = p_1(x_1', \xi_2)\,\mathrm{sgn}(\xi_2).$$

It is straightforward to show that the inverse Fourier transforms of these functions are given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} p_1(x_1', \xi_2)\,e^{-\mathrm{i}\xi_2 x_2'}\,\mathrm{d}\xi_2 = \frac{1}{2\sqrt{\pi}}\mathrm{Re}(x_1' + \mathrm{i}x_2')^{-3/2}, \tag{B.14}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} p_2(x_1', \xi_2)\,e^{-\mathrm{i}\xi_2 x_2'}\,\mathrm{d}\xi_2 = \frac{1}{2\sqrt{\pi}}\mathrm{Im}(x_1' + \mathrm{i}x_2')^{-3/2}. \tag{B.15}$$

Formulae (B.14), (B.15) enable one to evaluate $\langle\mathbf{U}\rangle_+$ in the form[10]

$$\langle\mathbf{U}\rangle_+(x_1', x_2') = \frac{C}{(-2\mathrm{i})^{1/2}}\frac{1}{2\sqrt{\pi}}\begin{bmatrix} 0 & 0 & -\mathrm{Re}\dfrac{(2-\nu)}{(x_1'+\mathrm{i}x_2')^{3/2}} \\ 0 & 0 & \mathrm{Im}\dfrac{(2-\nu)}{(x_1'+\mathrm{i}x_2')^{3/2}} \\ \mathrm{Re}\dfrac{2(1-\nu)}{(x_1'+\mathrm{i}x_2')^{3/2}} & -\mathrm{Im}\dfrac{2(1-\nu)}{(x_1'+\mathrm{i}x_2')^{3/2}} & 0 \end{bmatrix}$$

$$= \frac{1-2\nu}{8\sqrt{\pi}(2-\nu)(1-\nu)}\frac{1}{((x_1')^2+(x_2')^2)^{3/4}}\begin{pmatrix} 0 & 0 & -(2-\nu)\cos[\frac{3}{2}\mathscr{P}] \\ 0 & 0 & -(2-\nu)\sin[\frac{3}{2}\mathscr{P}] \\ 2(1-\nu)\cos[\frac{3}{2}\mathscr{P}] & 2(1-\nu)\sin[\frac{3}{2}\mathscr{P}] & 0 \end{pmatrix}, \tag{B.16}$$

where $\mathscr{P} = \tan^{-1}(x_2'/x_1')$.

As discussed after (B.5), comparison of (B.16) with Bueckner (1987) demonstrates agreement except (again) for the sign of the second column and, in addition, our (3, 2) entry corresponds to one half of that given by Bueckner.[11]

Evaluation of $\boldsymbol{I}_2$

The results of the above calculations can be used to analyze the term $\mathbf{I}_2$ in (3.26). Here we consider the case when the perturbation function ψ is independent of x_1 (in particular, we have such a perturbation in the example of the sinusoidal crack front presented in the main text). It follows from (3.14) that for negative values of x_1 the effective traction vector $[\boldsymbol{\sigma}_-^{(1)}]$ is given by

$$[\boldsymbol{\sigma}_-^{(1)}] = \frac{\partial}{\partial x_1}\begin{pmatrix} \psi(x_2)[\sigma_{11}^{(0)}]_-(x_1, x_2) \\ \psi(x_2)[\sigma_{12}^{(0)}]_-(x_1, x_2) \\ 0 \end{pmatrix} + \frac{\partial}{\partial x_2}\begin{pmatrix} \psi(x_2)[\sigma_{12}^{(0)}]_-(x_1, x_2) \\ \psi(x_2)[\sigma_{22}^{(0)}]_-(x_1, x_2) \\ 0 \end{pmatrix}, \tag{B.17}$$

where $[\sigma_{ij}^{(0)}]_-$ denotes the jump across the crack surface for the (i, j) stress component associated with the vector field $\mathbf{u}^{(0)}$. We suppose that the relations (3.15) hold everywhere on the crack surface, and then, when $x_1 < 0$, one has

[10] For the square root function of $z = re^{i\theta}$ we choose the branch $\sqrt{z} = r^{1/2}e^{\theta/2}$ with the branch cut along the non-negative half of the real axis.

[11] Since our method of derivation employs matrix manipulations and all other components agree, we believe that formula (B.16) is correct.

$$[\sigma_{11}^{(0)}]\ (x_1,x_2) = -(-2\pi x_1)^{-1/2}4K_{\mathrm{II}}^{(0)}(x_2),$$

$$[\sigma_{22}^{(0)}]\ (x_1,x_2) = -(-2\pi x_1)^{-1/2}4\nu K_{\mathrm{II}}^{(0)}(x_2),$$

$$[\sigma_{12}^{(0)}]\ (x_1,x_2) = -(-2\pi x_1)^{-1/2}2K_{\mathrm{III}}^{(0)}(x_2).$$

with ν being Poisson's ratio.

In our particular case the convolution $\langle \mathbf{U}\rangle'_+ * [\boldsymbol{\sigma}^{(1)}]$ admits the representation

$$(\langle \mathbf{U}\rangle'_+ * [\boldsymbol{\sigma}^{(1)}])(x'_1,x'_2) = -\frac{1-2\nu}{8\sqrt{2\pi}(1-\nu)}\begin{pmatrix} 0 \\ 0 \\ v(x'_1,x'_2) \end{pmatrix}, \tag{B.18}$$

where

$$v = 2\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} \mathrm{d}x_2 \int_{-\infty}^{0} \mathrm{d}x_1 \frac{\psi(x_2)K_{\mathrm{II}}^{(0)}(x_2)}{((x'_1-x_1)+\mathrm{i}(x'_2-x_2))^{3/2}(-x_1)^{3/2}}\right\}$$

$$-\mathrm{Im}\left\{\int_{-\infty}^{\infty} \mathrm{d}x_2 \int_{-\infty}^{0} \mathrm{d}x_1 \frac{\psi(x_2)K_{\mathrm{III}}^{(0)}(x_2)}{((x'_1-x_1)+\mathrm{i}(x'_2-x_2))^{3/2}(-x_1)^{3/2}}\right\}$$

$$+2\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} \mathrm{d}x_2 \int_{-\infty}^{0} \mathrm{d}x_1 \frac{\partial(\psi(x_2)K_{\mathrm{III}}^{(0)}(x_2))/\partial x_2}{((x'_1-x_1)+\mathrm{i}(x'_2-x_2))^{3/2}(-x_1)^{1/2}}\right\}$$

$$-4\nu\,\mathrm{Im}\left\{\int_{-\infty}^{\infty} \mathrm{d}x_2 \int_{-\infty}^{0} \mathrm{d}x_1 \frac{\partial(\psi(x_2)K_{\mathrm{II}}^{(0)}(x_2))/\partial x_2}{((x'_1-x_1)+\mathrm{i}(x'_2-x_2))^{3/2}(-x_1)^{1/2}}\right\}.$$

Note that first two integrals are singular, and are interpreted in terms of generalized functions.

The following relation holds

$$\int_{-\infty}^{0} \frac{\mathrm{d}x_1}{((x'_1-x_1)+\mathrm{i}(x'_2-x_2))^{3/2}(-x_1)^{1/2}} = \frac{2}{x'_1+\mathrm{i}(x'_2-x_2)}. \tag{B.19}$$

Also, the limit $x'_1 \to 0$ for a smooth function $q(x_2)$, which is bounded at infinity, one can write

$$\int_{-\infty}^{\infty} \frac{q(x_2)\,\mathrm{d}x_2}{x'_1+\mathrm{i}(x'_2-x_2)} \to \mathrm{i}P.V.\int_{-\infty}^{\infty} \frac{q(x_2)\,\mathrm{d}x_2}{x_2-x'_2} + \pi q(x'_2)\,\mathrm{sgn}(x'_1). \tag{B.20}$$

Here the Plemelj formulae were used. Thus, as $x'_1 \to +0$,

$$\int_{-\infty}^{\infty} \mathrm{d}x_2 \int_{-\infty}^{0} \mathrm{d}x_1 \frac{q(x_2)}{((x'_1-x_1)+\mathrm{i}(x'_2-x_2))^{3/2}(-x_1)^{1/2}} \to 2\mathrm{i}P.V.\int_{-\infty}^{\infty} \frac{q(x_2)\,\mathrm{d}x_2}{x_2-x'_2} + 2\pi q(x'_2). \tag{B.21}$$

Next, the convolutions involving the non-integrable singularity are deduced by noting that their correct interpretations in terms of generalized functions are obtained by differentiating (B.19) or (B.20) with respect to x'_1. Thus, considering only $x'_1 > 0$,

$$\int_{-\infty}^{0} \frac{\mathrm{d}x_1}{((x'_1-x_1)+\mathrm{i}(x'_2-x_2))^{3/2}(-x_1)^{3/2}}$$

$$= -3\int_{-\infty}^{0} \frac{\mathrm{d}x_1}{(-x_1)^{1/2}((x'_1-x_1)+\mathrm{i}(x'_2-x_2))^{5/2}} = -\frac{4}{(x'_1+\mathrm{i}(x'_2-x_2))^2}. \tag{B.22}$$

Further, for any smooth function $q(x_2)$ bounded at infinity

$$\int_{-\infty}^{\infty} \frac{q(x_2)\,\mathrm{d}x_2}{(x'_1+\mathrm{i}(x'_2-x_2))^2} = -\int_{-\infty}^{\infty} \frac{q'(x_2)\,\mathrm{d}x_2}{x_2-(x'_2-\mathrm{i}x_1)} \to P.V.\int_{-\infty}^{\infty} \frac{q'(x_2)\,\mathrm{d}x_2}{x_2-x'_2} - \pi\mathrm{i}q'(x'_2), \tag{B.23}$$

as $x_1 \to +0$ (again, we use the Plemelj formulae). Finally, it follows from (B.18) that

$$K_{\mathrm{I}}^{(\mathrm{skew})} = \lim_{x'_1\to+0}(\langle \mathbf{U}\rangle'_+ * [\boldsymbol{\sigma}^{(1)}])\mathbf{e}_3 = \frac{1-2\nu}{1-\nu}\frac{1}{\sqrt{2}}\left\{\frac{\partial}{\partial x'_2}(\psi K_{\mathrm{III}}^{(0)}) + \frac{(1-\nu)}{\pi}P.V.\int_{-\infty}^{\infty} \frac{\partial(\psi K_{\mathrm{II}}^{(0)})/\partial x_2}{x_2-x'_2}\mathrm{d}x_2\right\}, \tag{B.24}$$

which is consistent with (3.41).

APPENDIX C: STRESS FIELD NEAR THE CRACK FRONT. HIGH ORDER TERMS

Here present the derivation (based on analysis of the displacement field near the crack front) of explicit formulae for the vector coefficients **A** and **G** in the asymptotic expansion (3.22).

First by introducing local cylindrical coordinates (r, θ, z) such that

$$x_1 = r\cos\theta, \quad x_3 = r\sin\theta, \quad x_2 = -z,$$

we write the displacement vector in the following asymptotic form

$$\mathbf{u} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} \sim r^{1/2}\{K_{\rm I}(x_2)\mathbf{W}^{(1)}(\theta) + K_{\rm II}(x_2)\mathbf{W}^{(2)}(\theta) + K_{\rm III}(x_2)\mathbf{W}^{(3)}(\theta)\}$$

$$+\mathscr{L}(\mathbf{x}) + r^{3/2}\{a_1(x_2)\mathbf{V}^{(1)}(\theta) + a_2(x_2)\mathbf{V}^{(2)}(\theta) + a_3(x_2)\mathbf{V}^{(3)}(\theta)$$

$$+K'_{\rm I}(x_2)\boldsymbol{\Upsilon}^{(1)}(\theta) + K'_{\rm II}(x_2)\boldsymbol{\Upsilon}^{(2)}(\theta) + K'_{\rm III}(x_2)\boldsymbol{\Upsilon}^{(3)}(\theta)\} + O(r^2). \tag{C.1}$$

where $\mathscr{L}$ is a linear (with respect to r) vector function which produces constant stress field; the vector fields $\boldsymbol{\Upsilon}^{(i)}$ compensate discrepancies of high order in the Lamé system and traction boundary conditions left by the terms $r^{1/2}K_i\mathbf{W}^{(i)}$, $i = 1, 2, 3$.

The following notations are adopted

$$\mathbf{W}^{(1)}(\theta) = \begin{pmatrix} W_r^{(1)} \\ W_\theta^{(1)} \\ W_z^{(1)} \end{pmatrix} = \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{8\mu}\begin{bmatrix} -\cos\frac{3\theta}{2} + (2\kappa-1)\cos\frac{\theta}{2} \\ \sin\frac{3\theta}{2} - (2\kappa+1)\sin\frac{\theta}{2} \\ 0 \end{bmatrix}$$

$$\mathbf{W}^{(2)}(\theta) = \begin{pmatrix} W_r^{(2)} \\ W_\theta^{(2)} \\ W_z^{(2)} \end{pmatrix} = \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{8\mu}\begin{bmatrix} 3\sin\frac{3\theta}{2} - (2\kappa-1)\sin\frac{\theta}{2} \\ 3\cos\frac{3\theta}{2} - (2\kappa+1)\cos\frac{\theta}{2} \\ 0 \end{bmatrix}$$

$$\mathbf{W}^{(3)}(\theta) = \begin{pmatrix} W_r^{(3)} \\ W_\theta^{(3)} \\ W_z^{(3)} \end{pmatrix} = -\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\mu}\begin{bmatrix} 0 \\ 0 \\ \sin\frac{\theta}{2} \end{bmatrix},$$

$$\mathbf{V}^{(1)}(\theta) = \begin{bmatrix} \cos\frac{5\theta}{2} + (2\kappa-3)\cos\frac{\theta}{2} \\ -\sin\frac{5\theta}{2} + (2\kappa+3)\sin\frac{\theta}{2} \\ 0 \end{bmatrix},$$

$$\mathbf{V}^{(2)}(\theta) = \begin{bmatrix} 5\sin\frac{5\theta}{2} + (2\kappa-3)\sin\frac{\theta}{2} \\ 5\cos\frac{5\theta}{2} - (2\kappa+3)\cos\frac{\theta}{2} \\ 0 \end{bmatrix},$$

$$\mathbf{V}^{(3)}(\theta) = \begin{bmatrix} 0 \\ 0 \\ \sin\frac{3\theta}{2} \end{bmatrix};$$

as in the main text, $\kappa = 3-4\nu$, where ν is the Poisson ratio.

The standard representation for the Lamé operator in the cylindrical coordinate system is given by

$$\mathbf{L}^{(\mathrm{cyl})}\mathbf{u} = \left\{r^{-2}\mathbf{L}^{(0)}\left(r\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)+r^{-1}\mathbf{L}^{(1)}\left(r\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)+\mathbf{L}^{(2)}\frac{\partial^2}{\partial z^2}\right\}\mathbf{u},$$

where

$$\mathbf{L}^{(0)}(\xi,\eta) = \begin{pmatrix} (\xi^2-1)(\lambda+2\mu)+\mu\eta^2 & (\lambda+\mu)\xi\eta-(\lambda+3\mu)\eta & 0 \\ (\lambda+\mu)\xi\eta+(\lambda+3\mu)\eta & \mu(\xi^2-1)+(\lambda+2\mu)\eta^2 & 0 \\ 0 & 0 & \mu(\xi^2+\eta^2) \end{pmatrix},$$

$$\mathbf{L}^{(1)}(\xi,\eta) = (\lambda+\mu)\begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & \eta \\ \xi+1 & \eta & 0 \end{pmatrix},$$

$$\mathbf{L}^{(2)}(\xi,\eta) = \mathrm{diag}\,\{\mu,\mu,2\mu+\lambda\},$$

and for the operator of traction boundary conditions (for a half-plane crack) one has

$$\mathbf{B}^{(\mathrm{cyl})}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial\theta},\frac{\partial}{\partial z}\right)\mathbf{u} = \begin{pmatrix}\sigma_{r\theta}(\mathbf{u})\\ \sigma_{\theta\theta}(\mathbf{u})\\ \sigma_{z\theta}(\mathbf{u})\end{pmatrix} = \left\{r^{-1}\mathbf{B}^{(0)}\left(r\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)+\mathbf{B}^{(1)}\frac{\partial}{\partial z}\right\}\mathbf{u},$$

where

$$\mathbf{B}^{(0)}(\xi,\eta) = \begin{pmatrix} \mu\eta & \mu(\xi-1) & 0 \\ 2\mu+\lambda(\xi+1) & (2\mu+\lambda)\eta & 0 \\ 0 & 0 & \mu\eta \end{pmatrix},\quad \mathbf{B}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \mu & 0 \end{pmatrix}.$$

The vector functions $r^{3/2}\mathbf{\Upsilon}^{(j)}, j = 1,2,3$ from (C.1) solve the boundary value problems

$$r^{-2}\mathbf{L}^{(0)}\left(r\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)\{r^{3/2}\mathbf{\Upsilon}^{(j)}(\theta)\} = r^{-1}\mathbf{L}^{(1)}\left(r\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)\{r^{1/2}\mathbf{W}^{(j)}(\theta)\},$$

$$r^{-1}\mathbf{B}^{(0)}\left(r\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)\{r^{3/2}\mathbf{\Upsilon}^{(j)}(\theta)\}\Big|_{\theta=\pm\pi} = \mathbf{B}^{(1)}r^{1/2}\mathbf{W}^{(j)}(\pm\pi). \tag{C.2}$$

Direct calculations show that

$$\mathbf{\Upsilon}^{(1)} = \begin{pmatrix}\Upsilon_r^{(1)}\\ \Upsilon_\theta^{(1)}\\ \Upsilon_z^{(1)}\end{pmatrix} = \frac{1}{2\mu\sqrt{2\pi}}\begin{bmatrix} 0 \\ 0 \\ \cos\frac{\theta}{2}-\frac{1}{3}(2\kappa+1)\cos\frac{3\theta}{2}\end{bmatrix},$$

$$\mathbf{\Upsilon}^{(2)} = \begin{pmatrix}\Upsilon_r^{(2)}\\ \Upsilon_\theta^{(2)}\\ \Upsilon_z^{(2)}\end{pmatrix} = -\frac{1}{2\mu\sqrt{2\pi}}\begin{bmatrix} 0 \\ 0 \\ \sin\frac{\theta}{2}\end{bmatrix},$$

$$\mathbf{\Upsilon}^{(3)} = \begin{pmatrix}\Upsilon_r^{(3)}\\ \Upsilon_\theta^{(3)}\\ \Upsilon_z^{(3)}\end{pmatrix} = -\frac{2}{15\mu}\left(\frac{2}{\pi}\right)^{1/2}\begin{bmatrix} \kappa\sin\frac{\theta}{2} \\ (\kappa-1)\cos\frac{\theta}{2} \\ 0\end{bmatrix}.$$

When the crack is shifted in the x_3-direction by a small amount $\varepsilon\psi$, the vector of tractions, evaluated on the reference plane ahead of the crack front is characterised by the following asymptotic formula[12]

$$\begin{pmatrix}\sigma_{13}\\ \sigma_{23}\\ \sigma_{33}\end{pmatrix} \simeq \frac{1}{\sqrt{2\pi x_1}}\left\{\mathbf{I}-\frac{\varepsilon\psi}{2x_1}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\right\}\begin{pmatrix}K_{\mathrm{II}}\\ K_{\mathrm{III}}\\ K_{\mathrm{I}}\end{pmatrix}+\mathbf{P}+x_1^{1/2}\mathbf{A}-\frac{\varepsilon\psi}{2x_1^{1/2}}\mathbf{G}, \tag{C.3}$$

where $\mathbf{I}$ is the identity matrix, and $\mathbf{A}$ and $\mathbf{G}$ are given by

[12] In order to obtain (C.3) we evaluate components σ_{13}, σ_{23}, σ_{33} for the field (C.1), let $\theta = -\varepsilon\psi/x_1$ and expand to first order in ε (keeping x_1 fixed).

$$\mathbf{A} = \begin{bmatrix} 12\mu a_2 - \dfrac{1}{5}\left(\dfrac{2}{\pi}\right)^{1/2} K'_{\mathrm{III}}(x_2) \\ -\dfrac{3}{2}\mu a_3 - \dfrac{1}{8}(2\kappa - 3)\left(\dfrac{2}{\pi}\right)^{1/2} K'_{\mathrm{II}}(x_2) \\ 12\mu a_1 \end{bmatrix}, \tag{C.4}$$

$$\mathbf{G} = \begin{bmatrix} -12\mu a_1 \\ -\left(\dfrac{2}{\pi}\right)^{1/2} K'_{\mathrm{I}}(x_2) \\ -12\mu a_2 - \dfrac{4}{5}\left(\dfrac{2}{\pi}\right)^{1/2} K'_{\mathrm{III}}(x_2) \end{bmatrix}. \tag{C.5}$$

One can verify directly that formulae (C.4), (C.5) are consistent with (3.23).